Summarizing Variation Matrix Algebra & Mx

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Overview

- Mean/Variance/Covariance
  - Calculating
  - Estimating by ML
- Matrix Algebra
- Normal Theory Likelihood
- Mx script language

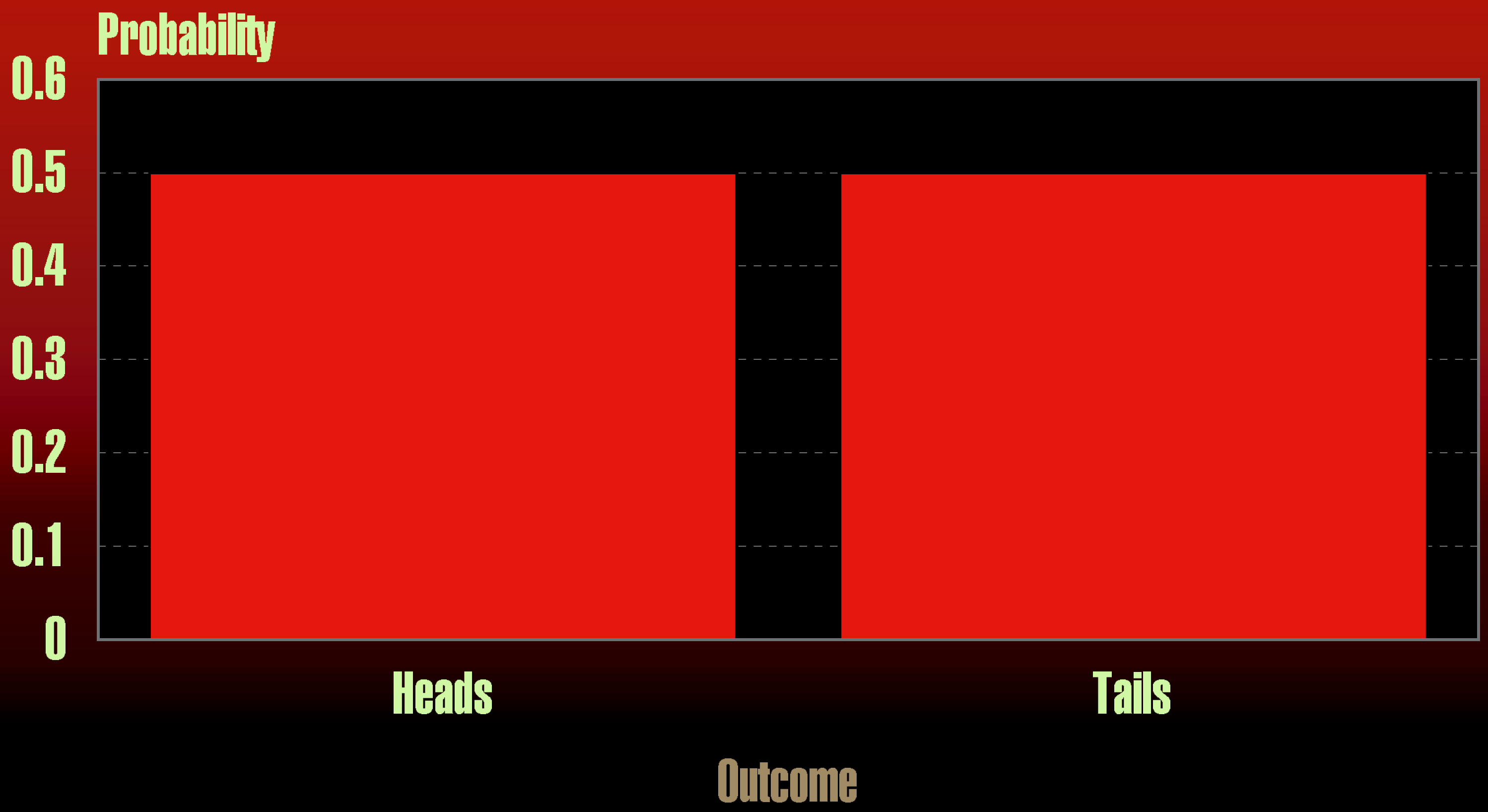
Computing Mean

- Formula $\sum(x_i)/N$

- Can compute with
  - Pencil
  - Calculator
  - SAS
  - SPSS
  - Mx

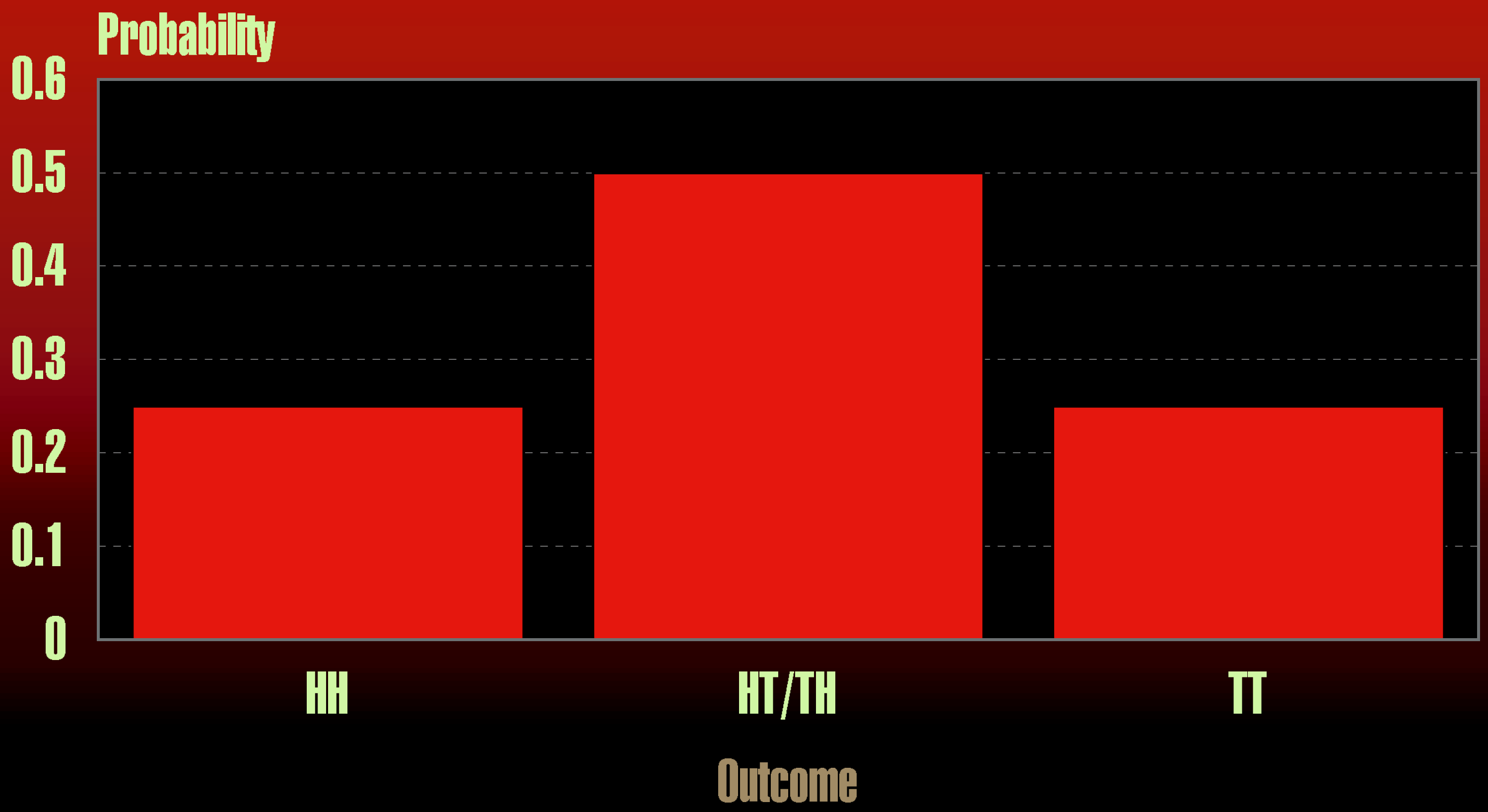
One Coin toss

2 outcomes



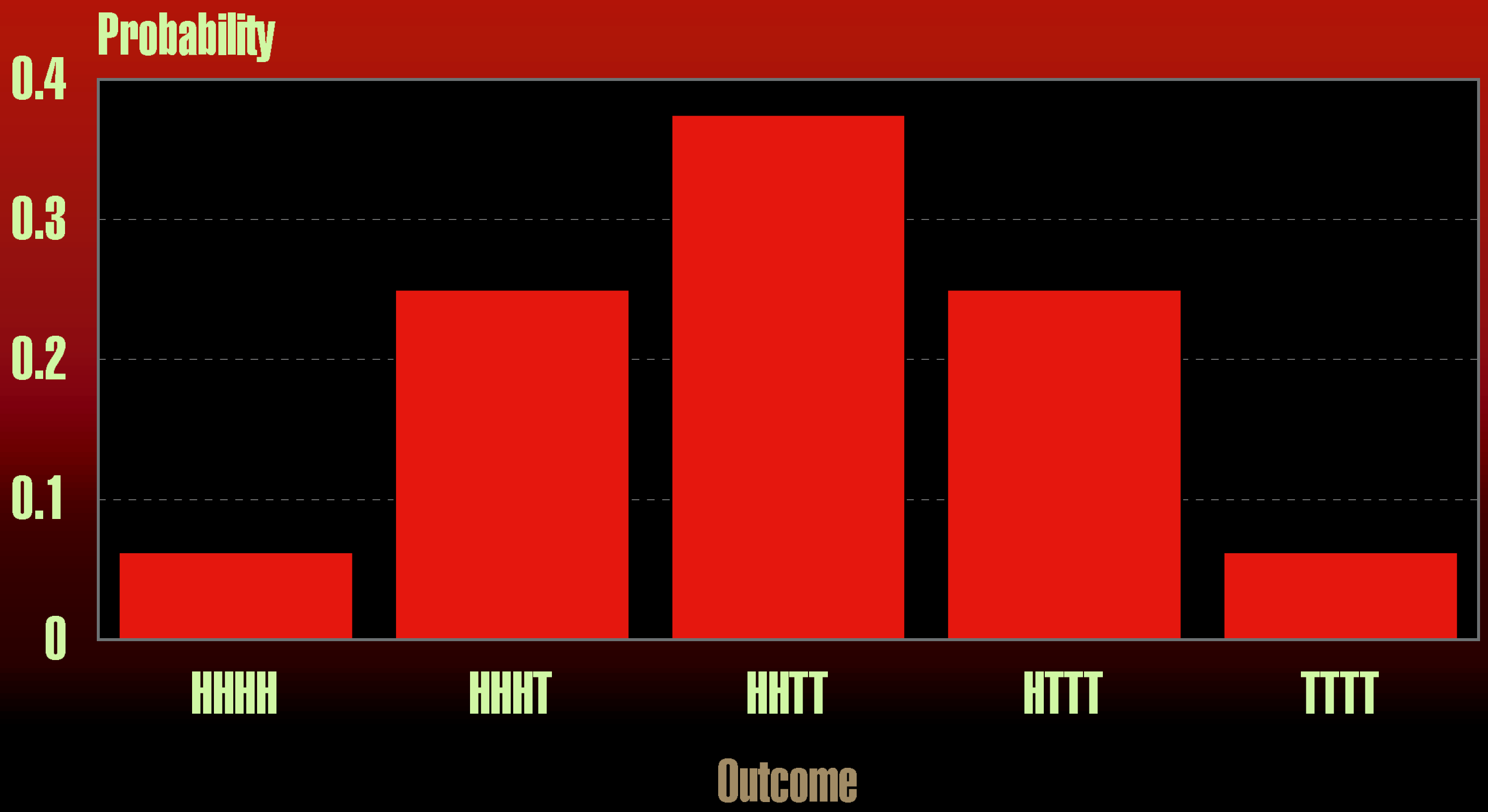
Two Coin toss

3 outcomes



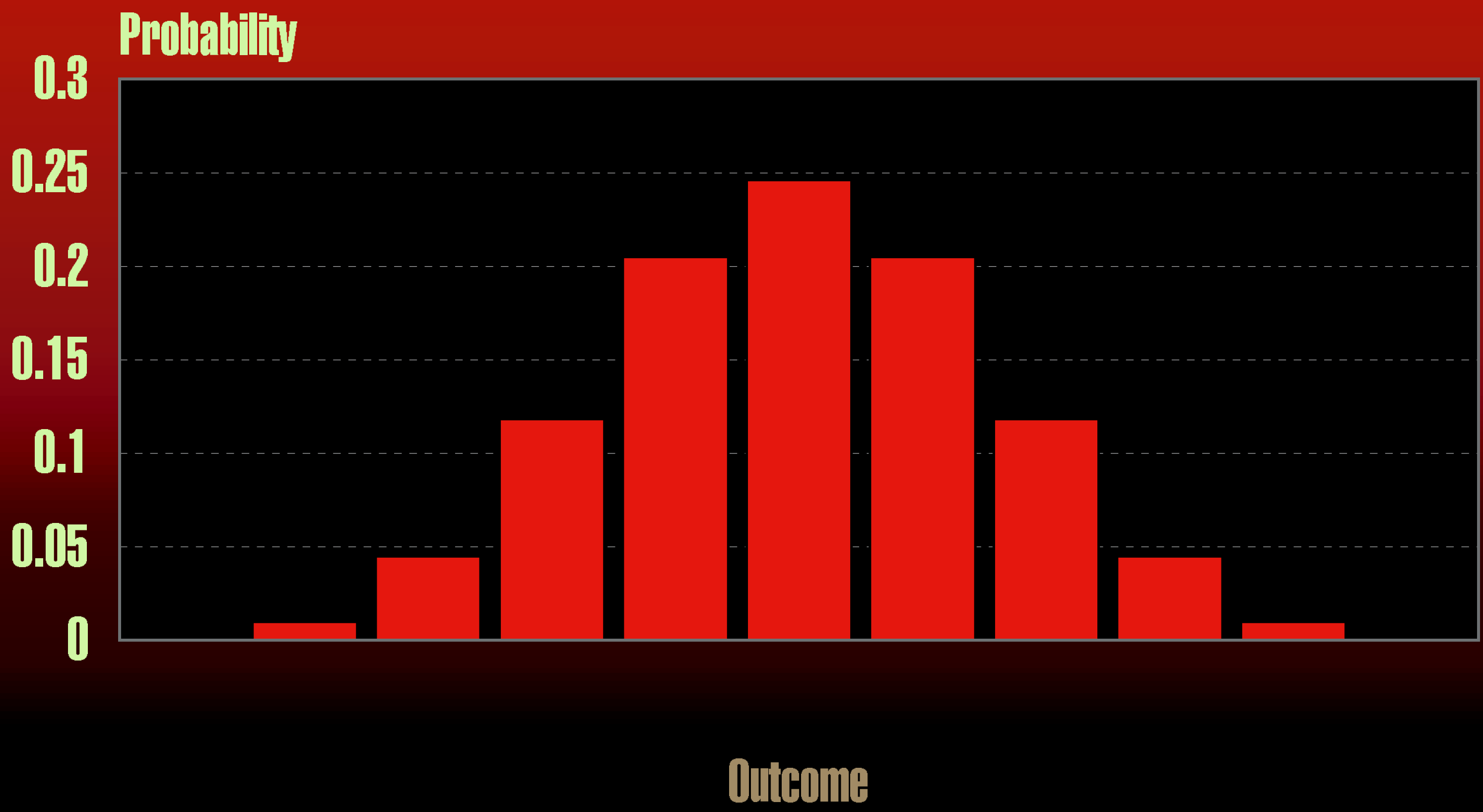
Four Coin toss

5 outcomes



Ten Coin toss

9 outcomes



Monty Python Theory

- Elk: The Theory by A. Elk brackets Miss brackets. My theory is along the following lines.
- Host: Oh God.
- Elk: All brontosauruses are thin at one end, much MUCH thicker in the middle, and then thin again at the far end.
- That is the theory that I have and which is mine, and what it is too.

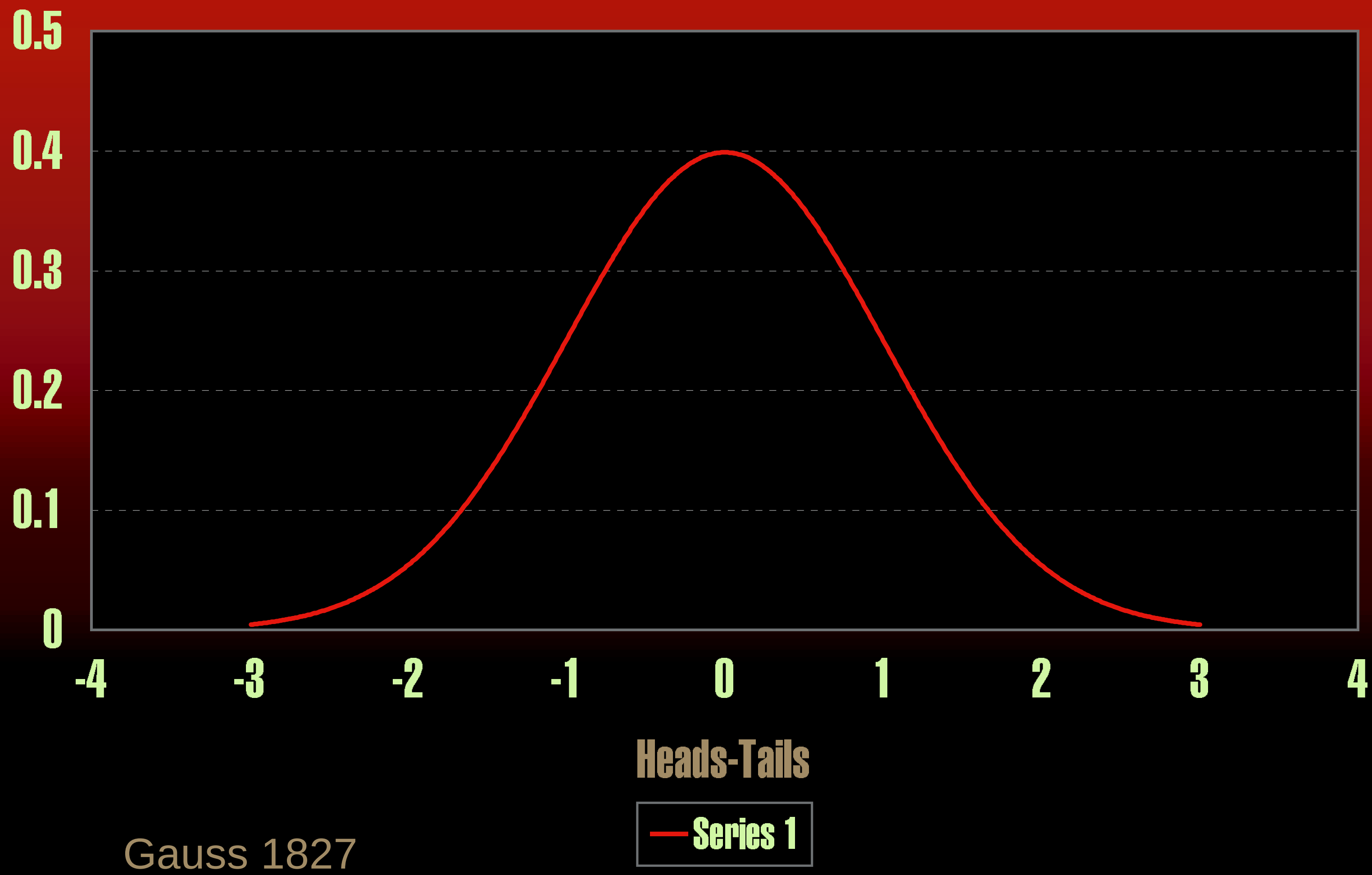
Pascal's Triangle

Frequency	Probability
1	1/1
1 1	1/2
1 2 1	1/4
1 3 3 1	1/8
1 4 6 4 1	1/16
1 5 10 10 5 1	1/32
1 6 15 20 15 6 1	1/64
1 7 21 35 35 21 7 1	1/128

Pascal's friend Chevalier de Mere 1654; Huygens 1657; Cardan 1501-1576

Fort Knox Toss

Infinite outcomes



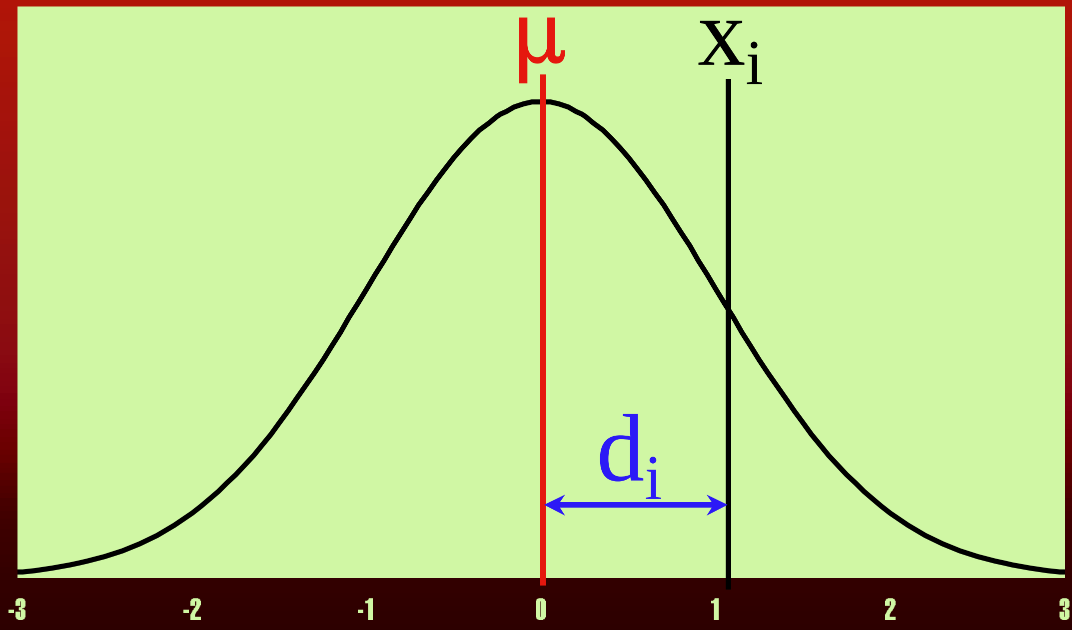
Gauss 1827

Variance

- Measure of Spread
- Easily calculated
- Individual differences

Average squared deviation

Normal distribution



$$\text{Variance} = \Sigma\ d_i^2/N$$

Measuring Variation

Weighs & Means

- Absolute differences?
- Squared differences?
- Absolute cubed?
- Squared squared?

Measuring Variation

Ways & Means

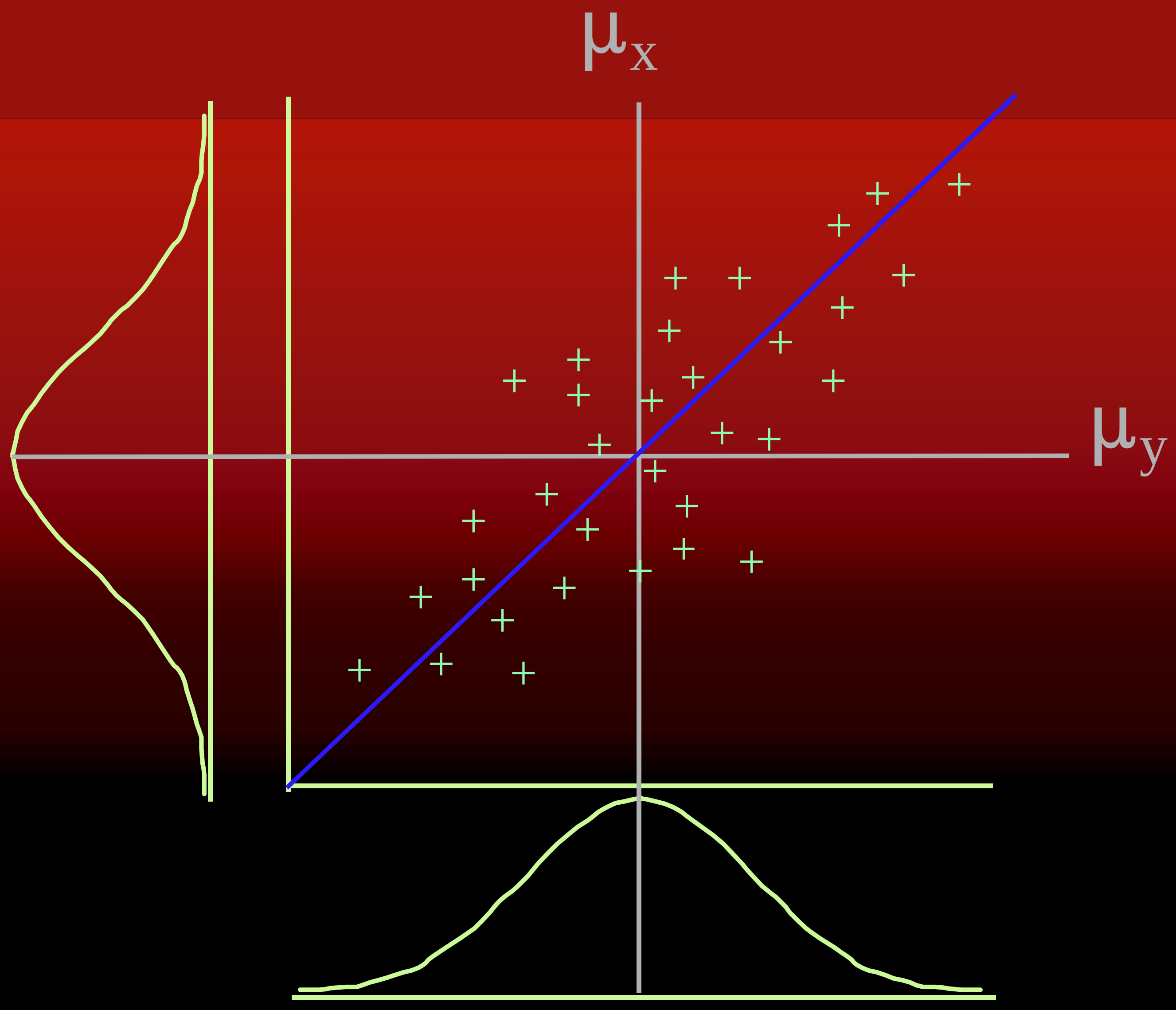
- Squared differences

Fisher (1922) Squared has minimum variance under normal distribution

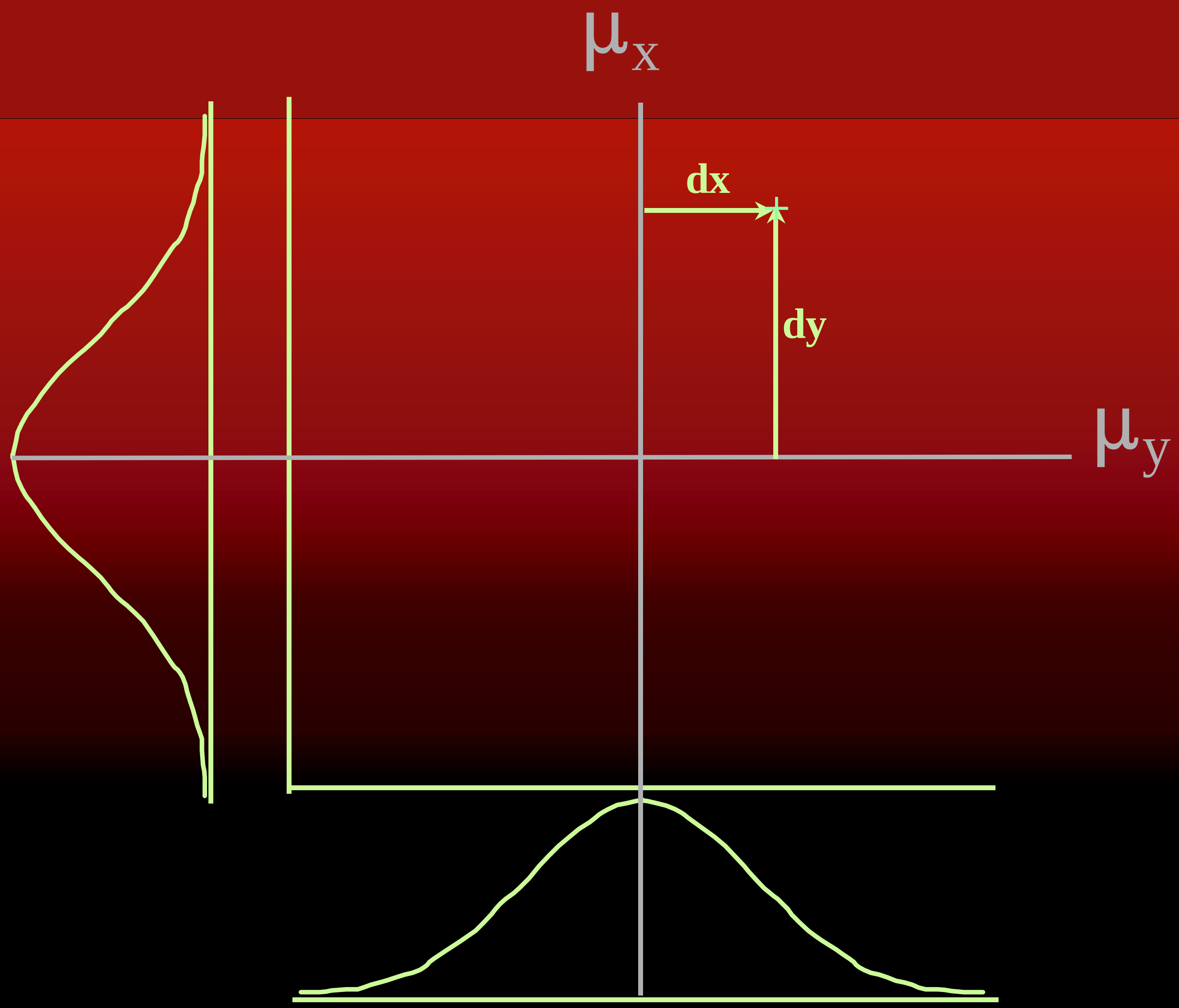
Covariance

- Measure of association between two variables
- Closely related to variance
- Useful to partition variance

Deviations in two dimensions



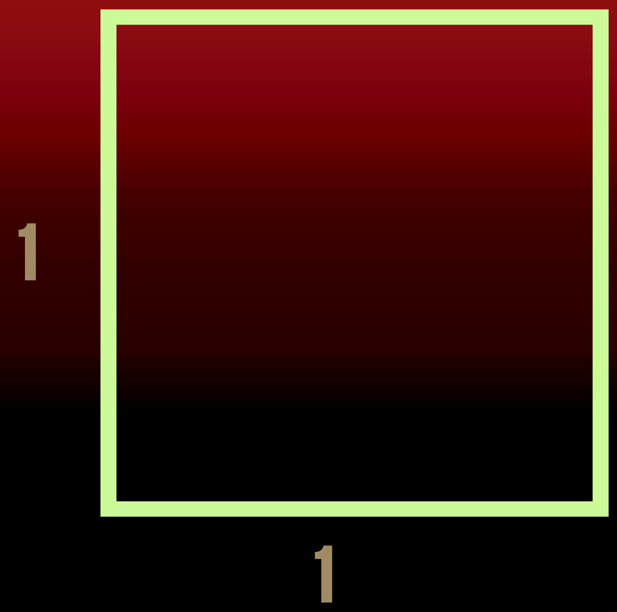
Deviations in two dimensions



Measuring Covariation

Area of a rectangle

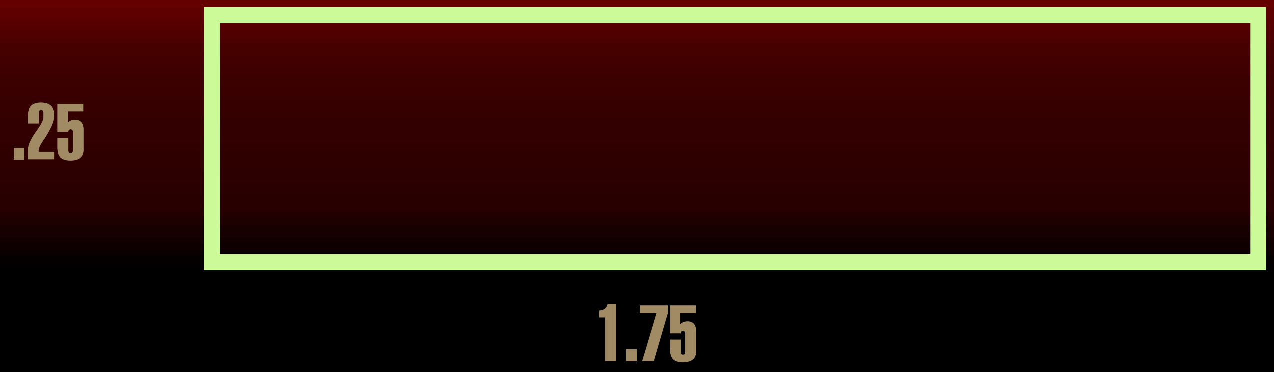
- A square, perimeter 4
- Area 1



Measuring Covariation

Area of a rectangle

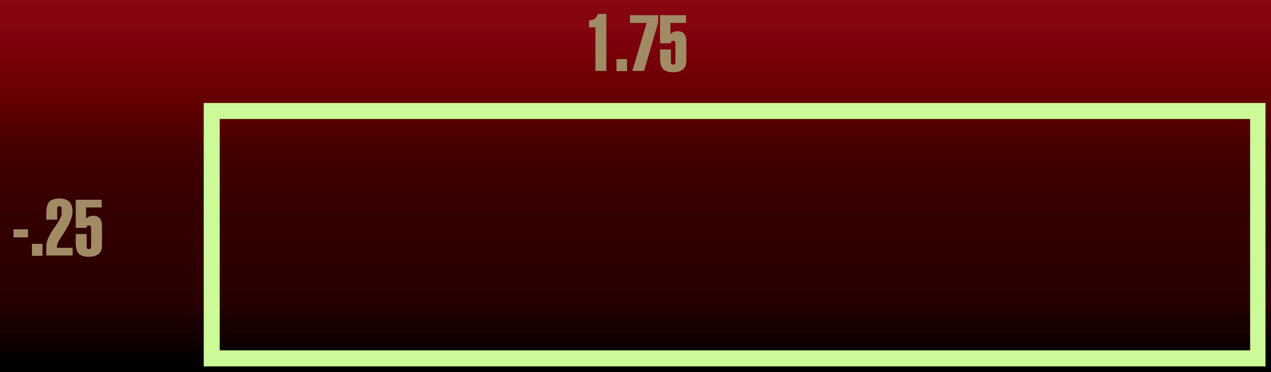
- A skinny rectangle, perimeter 4
- Area .25*1.75 = .4385



Measuring Covariation

Area of a rectangle

- Points can contribute negatively
- Area -.25*1.75 = -.4385



Measuring Covariation

Covariance Formula

$$\sigma_{xy} = \frac{\sum (x_i - \mu_x)(y_i - \mu_y)}{(N-1)}$$

Correlation

- Standardized covariance
- Lies between -1 and 1

$$r_{xy} = \frac{\sigma_{xy}}{\sqrt{\sigma_x^2 * \sigma_y^2}}$$

Summary

Formulae

$$\mu = (\Sigma x_i)/N$$

$$\sigma_x^2 = \Sigma(x_i - \mu)^2/(N-1)$$

$$\sigma_{xy} = \Sigma(x_i-\mu_x)(y_i-\mu_y)/(N-1)$$

$$r_{xy} = \frac{\sigma_{xy}}{\sqrt{\sigma_x^2 * \sigma_y^2}}$$

Variance covariance matrix

Several variables

$$\begin{bmatrix} Var(X) & Cov(X,Y) & Cov(X,Z) \\ Cov(X,Y) & Var(Y) & Cov(Y,Z) \\ Cov(X,Z) & Cov(Y,Z) & Var(Z) \end{bmatrix}$$

Variance covariance matrix

Univariate Twin Data

$$
\begin{bmatrix} Var(Twin1) & Cov(Twin1,Twin2) \\ Cov(Twin2,Twin1) & Var(Twin2) \end{bmatrix}
$$

Only suitable for complete data
Good conceptual perspective

Conclusion

- Means and covariances
- Conceptual underpinning
- Easy to compute
- Can use raw data instead

Model fitting to covariance matrices

- Inherently compares fit to saturated model
- Difference in fit between A C E model and A E model gives *likelihood ratio test*
- Asymptotically distributed as chi-squared with df = difference in number of parameters (df=1 for ACE vs. AE)

Estimate saturated model

Raw Data: Estimate means & covariances

- Need:
  - 1 Likelihood theory
  - 2 Data
  - 3 Estimate covariance matrix
  - 4 Estimate mean vector
  - 5 Program

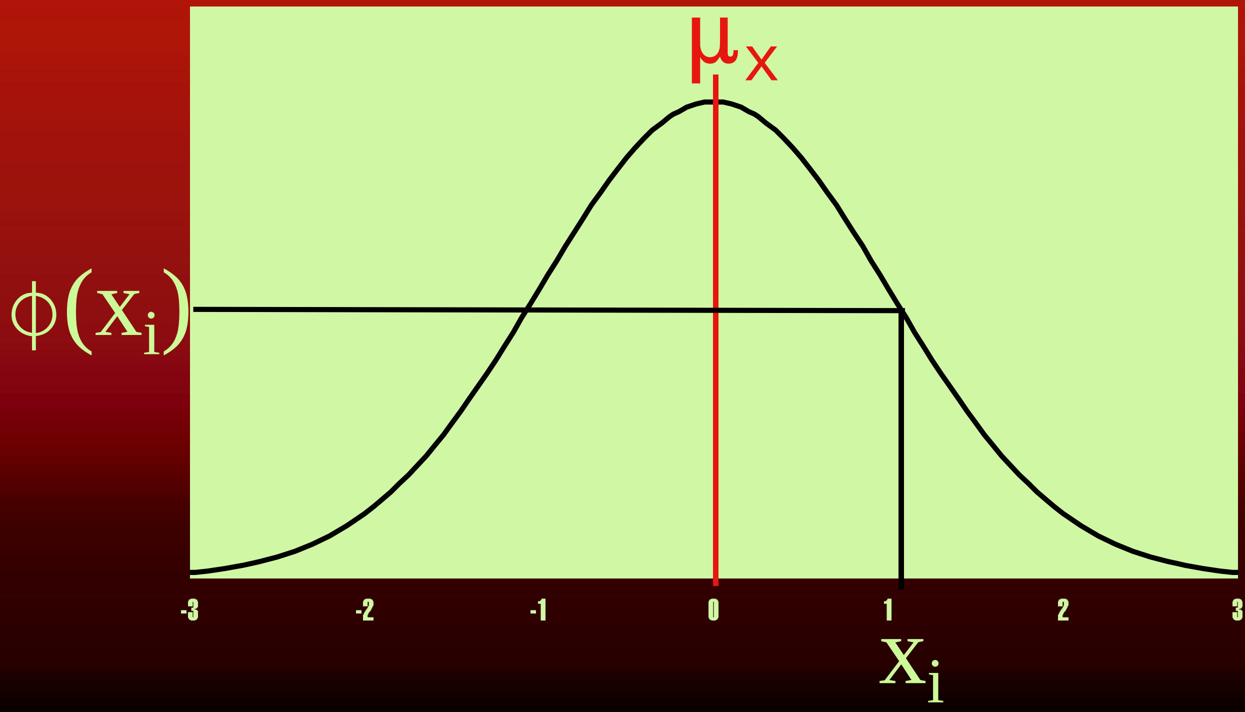
Likelihood computation

Calculate height of curve

- Univariate - height of normal pdf
  - $\phi(x) =$
  - $(2\pi\sigma^2)^{-.5}\ e^{-.5((x_i - \mu)^2)/\sigma^2)}$
- Multivariate - height of multinormal pdf
  - $|2\pi\Sigma|^{-n/2}\ e^{-.5((\mathbf{x_i} - \mu)\Sigma^{-1}(\mathbf{x_i} - \mu)')}$

Height of normal curve

Probability density function



Mx script to estimate covariances & means

```
#ngroups 1                                  !        mydatfile.dat e.g.
                                            !         Data NI=2
#define nvar=2                              !         Labels BP-T1 BP-T2
                                            !         Rectangular File=mzbp.rec
G1: Estimate means & Covariances by ML
 #include mydatfile.dat

 Begin Matrices;                            !        mzbp.rec e.g.
  C Symm nvar nvar  Free                    !          120.5 142.3
  M Full    1 nvar  Free                    !          102.6 110.7
 End Matrices;                              !           98.3 116.9

  Matrix C 1 0 1                            !        starting values for C
  Covariance C;
  Means M;

 Option NDecimals=2
End
```

Matrix Algebra

- You already know a lot of it
- Economical and aesthetic
- Great for statistics

What you know

All about (1x1) matrices

Operation	Example	Result
Addition	2 + 2	
Subtraction	5 - 1	
Multiplication	2 x 2	
Division	12 / 3	

What you know

All about (1x1) matrices

Operation	Example	Result
Addition	2 + 2	4
Subtraction	5 - 1	4
Multiplication	2 x 2	4
Division	12 / 3	4

What you may guess

Numbers can be organized
in boxes, e.g.

1	2
3	4

What you may guess

Numbers can be organized in boxes, e.g.

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

Matrix notation

$\underline{A}$

Many numbers

31 23 16 99 08 12 14 73 85 98 33 94 12 75 02 57 92 75 11
28 39 57 17 38 18 38 65 10 73 16 73 77 63 18 56 18 57 02
74 82 20 10 75 84 19 47 14 11 84 08 47 57 58 49 48 28 42
88 84 47 48 43 05 61 75 98 47 32 98 15 49 01 38 65 81 68
43 17 65 21 79 43 17 59 41 37 59 43 17 97 65 41 35 54 44
75 49 03 86 93 41 76 73 19 57 75 49 27 59 34 27 59 34 82
43 19 74 32 17 43 92 65 94 13 75 93 41 65 99 13 47 56 34
75 83 47 48 73 98 47 39 28 17 49 03 63 91 40 35 42 12 54
31 87 49 75 48 91 37 59 13 48 75 94 13 75 45 43 54 32 53
75 48 90 37 59 37 59 43 75 90 33 57 75 89 43 67 74 73 10
34 92 76 90 34 17 34 82 75 98 34 27 69 31 75 93 45 48 37
13 59 84 76 59 13 47 69 43 17 91 34 75 93 41 75 90 74 17
34 15 74 91 35 79 57 42 39 57 49 02 35 74 23 57 75 11 35

Matrix notation

$\underline{A}$

Useful subnotation

$${}_{2}\underline{A}_{2}$$

Useful subnotation

$$\underline{A}_{8\ \ 40}$$

Matrix operations

Addition

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} =$$

A + B =

Matrix operations

Addition

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 6 & 8 \\ 10 & 12 \end{bmatrix}$$

<u>A</u> + <u>B</u> = <u>C</u>

Matrix operations

Conformability

Addition

To add two matrices A and B:

# of rows in A = # of rows in B

# of columns in A = # of columns in B

Matrix operations

Subtraction

$$\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} =$$

$\underline{B} - \underline{A} =$

Matrix operations

Subtraction

$$\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix}$$

$\underline{B} - \underline{A} = \underline{C}$

Matrix operations

Conformability

Subtraction

To subtract two matrices A and B:

# of rows in A = # of rows in B

# of columns in A = # of columns in B

Matrix operations

Multiplication (regular)

Conformability

To multiply two matrices A and B:

# of columns in A = # of rows in B

Matrix multiplication

Multiply: A (m x n) by B (n by p)

$$\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} \times \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix}$$

A x B = C

General Formula

$$C_{ij} = \sum_{k=1}^{n} A_{ik} \times B_{kj}$$

Matrix multiplication

Multiply: A (m x n) by B (n by p)

$$\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} \times \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} (5x1)+(6x3) & ? \\ ? & ? \end{bmatrix}$$

A x B = C

General Formula

$$C_{ij} = \sum_{k=1}^{n} A_{ik} \times B_{kj}$$

Matrix multiplication

Multiply: A (m x n) by B (n by p)

$$\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} \times \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} (5x1)+(6x3) & (5x2)+(6x4) \\ ? & ? \end{bmatrix}$$

A x B = C

General Formula

$$C_{ij} = \sum_{k=1}^{n} A_{ik} \times B_{kj}$$

Matrix multiplication

Multiply: A (m x n) by B (n by p)

$$\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} \times \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} (5x1)+(6x3) & (5x2)+(6x4) \\ (7x1)+(8x3) & ? \end{bmatrix}$$

A x B = C

General Formula

$$C_{ij} = \sum_{k=1}^{n} A_{ik} \times B_{kj}$$

Matrix multiplication

Multiply: A (m x n) by B (n by p)

$$\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} \times \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} (5x1)+(6x3) & (5x2)+(6x4) \\ (7x1)+(8x3) & (7x2)+(8x4) \end{bmatrix}$$

A x B = C

General Formula

$$C_{ij} = \sum_{k=1}^{n} A_{ik} \times B_{kj}$$

Matrix multiplication

Multiply: A (m x n) by B (n by p)

$$\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} \times \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 23 & 34 \\ 31 & 46 \end{bmatrix}$$

A x B = C

General Formula

$$C_{ij} = \sum_{k=1}^{n} A_{ik} \times B_{kj}$$

Inner product of a vector

(Column) Vector c (n x 1)

eg n = 3

$$\begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix}$$

Inner product is c'c:

$$\begin{bmatrix} 2 & 4 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix} = (2 \times 2) + (4 \times 4) + (1 \times 1)$$

$$= 21$$

(Sum of squares)

Outer product of a vector

(Column) Vector c (n x 1)

eg n = 3

$$\begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix}$$

Outer product is cc'

$$\begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix} [2 \quad 4 \quad 1] = \begin{matrix} 2x2 & 2x4 & 2x1 \\ 4x2 & 4x4 & 4x1 \\ 1x2 & 1x4 & 1x1 \end{matrix}$$

$$= \begin{matrix} 4 & 8 & 2 \\ 8 & 16 & 4 \\ 2 & 4 & 1 \end{matrix}$$

Matrix Algebra

EXERCISES !

Unary operations: Inverse

A number can be divided by another number -

How do you divide matrices?

Note that a / b = $a \times \frac{1}{b}$

And that $a \times \frac{1}{a} = 1$

$\frac{1}{a}$ is the *inverse* of a

Unary operations: Inverse

Matrix 'equivalent' of 1 is the identity matrix

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Find A^{-1} such that $A^{-1} A = I$

Unary operations: Inverse

Inverse of (2 x 2) matrix

1. Find determinant
2. Swap a11 and a22
3. Change signs of a12 and a21
4. Divide each element by determinant
5. Check by pre- or post- multiplying by inverse

Unary operations: Inverse

Inverse of (2 x 2) matrix

1. Find determinant

= (a11 x a22) - (a21 x a12)

e.g. $A = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$

det(A) = (2x3) - (1x5) = 1

Unary operations: Inverse

Inverse of (2 x 2) matrix

1. Find determinant ✓
2. Swap a11 and a22
3. Change signs of a12 and a21
4. Divide each element by determinant
5. Check by pre- or post- multiplying by inverse

Unary operations: Inverse

Inverse of (2 x 2) matrix

2. Swap elements a11 and a22

e.g. $A = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$

becomes $\begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$

Unary operations: Inverse

Inverse of (2 x 2) matrix

√
√

1. Find determinant
2. Swap a11 and a22
3. Change signs of a12 and a21
4. Divide each element by determinant
5. Check by pre- or post- multiplying by inverse

Unary operations: Inverse

Inverse of (2 x 2) matrix

3. Change sign of a12 and a21

e.g. $A = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$

becomes $\begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$

Unary operations: Inverse

Inverse of (2 x 2) matrix

1. Find determinant √
2. Swap a11 and a22 √
3. Change signs of a12 and a21 √
4. Divide each element by determinant
5. Check by pre- or post- multiplying by inverse

Unary operations: Inverse

Inverse of (2 x 2) matrix

4. Divide each element by determinant

e.g. $A = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$

becomes $\begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$

Unary operations: Inverse

Inverse of (2 x 2) matrix

1. Find determinant √
2. Swap a11 and a22 √
3. Change signs of a12 and a21 √
4. Divide each element by determinant √
5. Check by pre- or post- multiplying by inverse

Unary operations: Inverse

Inverse of (2 x 2) matrix

5. Check result with $A \, A^{-1} = I$

e.g. $$\begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$$

equals $$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Matrix Algebra

EXERCISES !

Matrices can have special forms

Scalar (1 x 1) [9]

Row vector (1 x n) [2 6 3 8]

Column vector (n x 1)

$$\begin{bmatrix} 1 \\ 5 \\ 7 \\ 2 \end{bmatrix}$$

Matrix notation

(column) vector

$$\underline{a}$$

Matrix notation

(row) vector

$$\underline{a}'$$

More special matrices

Square (n x n)

3	12	18
15	8	21
9	17	86

Symmetric (n x n)

72	8	10
8	14	7
10	7	44

Diagonal (n x n)

19	0	0
0	12	0
0	0	17

Even more special matrices

Lower Triangular (n x n)

3	0	0
15	8	0
9	17	86

Sub-diagonal (n x n)

0	0	0
8	0	0
10	7	0

Standardized (n x n)

1	.5	.3
.5	1	.2
.3	.2	1

Yet more special matrices

Identity (nxn)

1	0	0	0
0	1	0	0
0	0	1	0
0	0	0	1

Full (mxn)

23	18	15	8
7	13	80	72

Matrix names

$$\underline{B}$$

Single letter in Mx