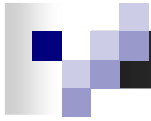


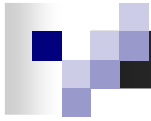
Matrix Algebra

International Workshop on
Methodology for Genetic Studies
Boulder Colorado March 2006



Heuristic or Horrific?

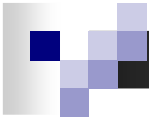
- You already know a lot of it
- Economical and aesthetic
- Great for statistics



What you most likely know

- All about (1x1) matrices

■ Operation	Example	Result
■ Addition	$2 + 2$	
■ Subtraction	$5 - 1$	
■ Multiplication	2×2	
■ Division	$12 / 3$	



What you most likely know

- All about (1x1) matrices

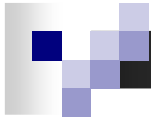
■ Operation	Example	Result
■ Addition	$2 + 2$	4
■ Subtraction	$5 - 1$	4
■ Multiplication	2×2	4
■ Division	$12 / 3$	4



What you may guess

- Numbers can be organized in boxes, e.g.

1	2
3	4



What you may guess

- Numbers can be organized in boxes, e.g.

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$



Matrix Notation

A



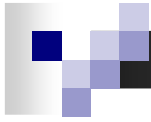
Many Numbers

31 23 16 99 08 12 14 73 85 98 33 94 12 75 02 57 92 75 11
28 39 57 17 38 18 38 65 10 73 16 73 77 63 18 56 18 57 02
74 82 20 10 75 84 19 47 14 11 84 08 47 57 58 49 48 28 42
88 84 47 48 43 05 61 75 98 47 32 98 15 49 01 38 65 81 68
43 17 65 21 79 43 17 59 41 37 59 43 17 97 65 41 35 54 44
75 49 03 86 93 41 76 73 19 57 75 49 27 59 34 27 59 34 82
43 19 74 32 17 43 92 65 94 13 75 93 41 65 99 13 47 56 34
75 83 47 48 73 98 47 39 28 17 49 03 63 91 40 35 42 12 54
31 87 49 75 48 91 37 59 13 48 75 94 13 75 45 43 54 32 53
75 48 90 37 59 37 59 43 75 90 33 57 75 89 43 67 74 73 10
34 92 76 90 34 17 34 82 75 98 34 27 69 31 75 93 45 48 37
13 59 84 76 59 13 47 69 43 17 91 34 75 93 41 75 90 74 17
34 15 74 91 35 79 57 42 39 57 49 02 35 74 23 57 75 11 35



Matrix Notation

A



Useful Subnotation

$A_{2 \quad 2}$



Useful Subnotation

A

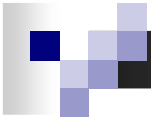
8

40



Matrix Operations

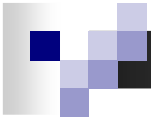
- Addition
- Subtraction
- Multiplication
- Inverse



Addition

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} & \\ & \end{bmatrix}$$

$$\underline{\mathbf{A}} + \underline{\mathbf{B}} =$$



Addition

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 6 & 8 \\ 10 & 12 \end{bmatrix}$$

A

+

B

=

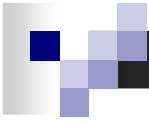
C



Addition Conformability

To add two matrices A and B :

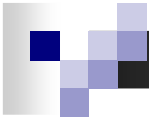
- # of rows in A = # of rows in B
- # of columns in A = # of columns in B



Subtraction

$$\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} & \\ & \end{bmatrix}$$

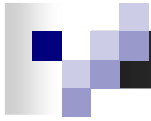
$$\underline{\mathbf{B}} - \underline{\mathbf{A}} =$$



Subtraction

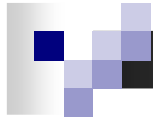
$$\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix}$$

$$\underline{\mathbf{B}} - \underline{\mathbf{A}} = \underline{\mathbf{C}}$$



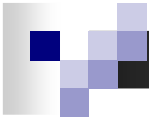
Subtraction Conformability

- To subtract two matrices A and B :
- # of rows in A = # of rows in B
- # of columns in A = # of columns in B



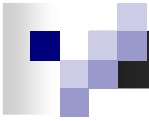
Multiplication Conformability

- Regular Multiplication
- To multiply two matrices A and B:
- # of columns in A = # of rows in B
- Multiply: A ($m \times n$) by B (n by p)



Multiplication General Formula

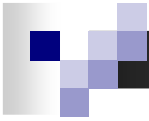
$$C_{ij} = \sum_{k=1}^n A_{ik} \times B_{kj}$$



Multiplication I

$$\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} \times \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} & \\ & \end{bmatrix}$$

$$\underline{\mathbf{A}} \times \underline{\mathbf{B}} =$$

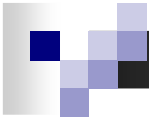


Multiplication II

$$\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} \times \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} (5 \times 1) \\ \end{bmatrix}$$

$$\underline{\mathbf{A}} \times \underline{\mathbf{B}} = \underline{\mathbf{C}}$$

$$C_{11} = \sum_{k=1}^n A_{1k} \times B_{k1}$$

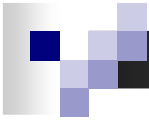


Multiplication III

$$\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} \times \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} (5 \times 1) + (6 \times 3) & \\ & \end{bmatrix}$$

$$\underline{\mathbf{A}} \times \underline{\mathbf{B}} = \underline{\mathbf{C}}$$

$$C_{11} = \sum_{k=2}^n A_{1k} \times B_{k1}$$

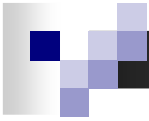


Multiplication IV

$$\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} \times \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 23 & (5 \times 2) + (6 \times 4) \end{bmatrix}$$

$$\underline{\mathbf{A}} \times \underline{\mathbf{B}} = \underline{\mathbf{C}}$$

$$C_{12} = \sum_{k=1}^n A_{1k} \times B_{k2}$$



Multiplication V

$$\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} \times \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 23 & 34 \\ (7 \times 1) + (8 \times 3) & \end{bmatrix}$$

$$\underline{\mathbf{A}} \times \underline{\mathbf{B}} = \underline{\mathbf{C}}$$

$$C_{21} = \sum_{k=1}^n A_{2k} \times B_{k1}$$

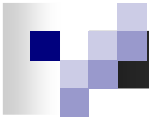


Multiplication VI

$$\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} \times \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 23 & 34 \\ 31 & (7 \times 2) + (8 \times 4) \end{bmatrix}$$

$$\underline{\mathbf{A}} \times \underline{\mathbf{B}} = \underline{\mathbf{C}}$$

$$C_{22} = \sum_{k=1}^n A_{2k} \times B_{k2}$$



Multiplication VII

$$\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} \times \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 23 & 34 \\ 31 & 46 \end{bmatrix}$$

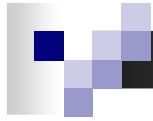
A
m x n

x

B
n x p

=

C
m x p



Transpose

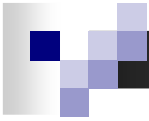
- Usually denoted by ' '
- Sometimes T

- Exchanges rows and columns
- $(m \times n)$ matrix becomes $(n \times m)$
- $A_{ij} = A_{ji}$

Inner Product of a Vector

■ (Column) Vector \mathbf{c} ($n \times 1$) $\underline{\mathbf{C}} = \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix}$

$$\begin{aligned} \begin{bmatrix} 2 & 4 & 1 \end{bmatrix} \times \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix} &= \begin{bmatrix} (2 \times 2) + (4 \times 4) + (1 \times 1) \end{bmatrix} \\ \underline{\mathbf{C}}' &= \underline{\mathbf{C}} = \underline{\mathbf{C}}' \underline{\mathbf{C}} \end{aligned}$$

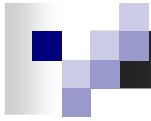


Outer Product of a Vector

■ (Column) vector \mathbf{c} ($n \times 1$) $\underline{\mathbf{C}} = \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix}$

$$\begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix} \times \begin{bmatrix} 2 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 8 & 2 \\ 8 & 16 & 4 \\ 2 & 4 & 1 \end{bmatrix}$$

$\underline{\mathbf{C}} \quad \underline{\mathbf{C}}' \quad \underline{\mathbf{C}} \quad \underline{\mathbf{C}}'$



Inverse

- A number can be divided by another number -
How do you divide matrices?
- Note that $a / b = a \times 1 / b$
- And that $a \times 1 / a = 1$
- $1 / a$ is the inverse of a

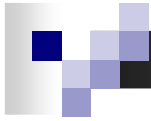


Unary operations: Inverse

- Matrix 'equivalent' of 1 is the identity matrix

$$\underline{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

- Find A^{-1} such that $A^{-1} * A = I$



Unary Operations: Inverse

- Inverse of (2 x 2) matrix
 - Find determinant
 - Swap a_{11} and a_{22}
 - Change signs of a_{12} and a_{21}
 - Divide each element by determinant
 - Check by pre- or post-multiplying by inverse



Inverse of 2 x 2 matrix

- Find the determinant
= $(a_{11} \times a_{22}) - (a_{21} \times a_{12})$
For

$$\underline{\mathbf{A}} = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$$

$$\det(\mathbf{A}) = (2 \times 3) - (1 \times 5) = 1$$



Inverse of 2 x 2 matrix

- Swap elements a_{11} and a_{22}

Thus

$$\underline{\mathbf{A}} = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$$

becomes

$$\begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$$



Inverse of 2 x 2 matrix

- Change sign of a_{12} and a_{21}

Thus

$$\underline{\mathbf{A}} = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$$

becomes

$$\begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$$



Inverse of 2 x 2 matrix

- Divide every element by the determinant

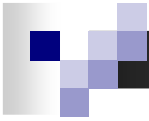
Thus

$$\underline{\mathbf{A}} = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$$

becomes

$$\begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$$

(luckily the determinant was 1)



Inverse of 2 x 2 matrix

- Check results with $\mathbf{A}^{-1} \mathbf{A} = \mathbf{I}$

Thus

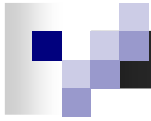
$$\begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix} \times \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$$

equals

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

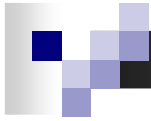


Intro to Mx Script Language



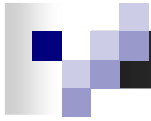
General Comments

- case insensitive, except for filenames under Unix
- comments: anything following a !
- blank lines
- commands: identified by first 2 letters, BUT recommended to use full words



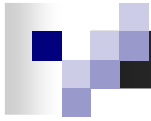
Job Structure

- three types of groups:
 - Data, Calculation, Constraint
- number of groups indicated by
 - #NGroups 3
 - at the beginning of job
- jobs can be stacked in one run



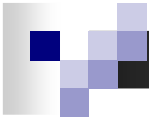
Group Structure

- Title
- Group type: data, calculation, constraint
 - [Read observed data, Select, Labels]
- Matrices declaration
 - [Specify numbers, parameters, etc.]
- Algebra section and/or Model statement
 - [Options]
- End



Read Observed Data

- Data NInputvars=2 [NObservations=123]
- CMatrix/ Means/ CTable/
 - summary statistics
 - read from script / file (File=filename)
- Rectangular/ Ordinal / VLength
 - raw data
 - read from script / file (File=filename)
- Select variables ; [by number/label]
- Labels variables



Matrix Declaration

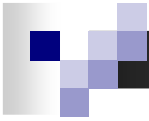
- Group 1
- Begin Matrices;
 - C Full 2 3 Free ! [name type rows columns free]
 - ! more matrices ! default element is fixed at 0
- End Matrices;

- Group 2
- Begin Matrices = Group 1;
 - ! copies all matrices from group 1
 - D Full 2 3 = C1 ! equates D to C of group 1



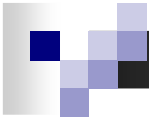
Matrix Types (Mx manual p.56)

Type	Structure	Shape	Free
Zero	Null (zeros)	Any	0
Unit	Unit (ones)	Any	0
Iden	Identity	Square	0
Diag	Diagonal	Square	r
S Diag	Subdiagonal	Square	$r(r-1)/2$
Stand	Standardized	Square	$r(r-1)/2$
Symm	Symmetric	Square	$r(r+1)/2$
Lower	Lower triangular	Square	$r(r+1)/2$
Full	Full	Any	$r \times c$
Computed	Equated to	Any	0



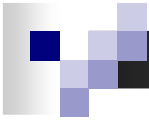
Matrices

Example Command	Specification Matrix	Values
A Zero 2 3 Free	0 0 0 0 0 0	0 0 0 0 0 0
B Unit 2 3 Free	0 0 0 0 0 0	1 1 1 1 1 1
C Iden 3 3 Free	0 0 0 0 0 0 0 0 0	1 0 0 0 1 0 0 0 1
D Izero 2 5 Free	0 0 0 0 0 0 0 0 0 0	1 0 0 0 0 0 1 0 0 0
E Ziden 2 5 Free	0 0 0 0 0 0 0 0 0 0	0 0 0 1 0 0 0 0 0 1



Matrices II

Example Command	Specification Matrix	Values
F Diag 3 3 Free	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$	$\begin{bmatrix} ? & 0 & 0 \\ 0 & ? & 0 \\ 0 & 0 & ? \end{bmatrix}$
G SDiag 3 3 Free	$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 2 & 3 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 0 \\ ? & 0 & 0 \\ ? & ? & 0 \end{bmatrix}$
H Stand 3 3 Free	$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 2 & 3 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & ? & ? \\ ? & 1 & ? \\ ? & ? & 1 \end{bmatrix}$



Matrices III

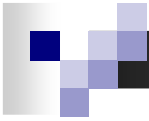
Example Command	Specification Matrix	Values
I Symm 3 3 Free	1 2 4 2 3 5 4 5 6	? ? ? ? ? ? ? ? ?
J Lower 3 3 Free	1 0 0 2 3 0 4 5 6	? 0 0 ? ? 0 ? ? ?
K Full 2 4 Free	1 2 3 4 5 6 7 8	? ? ? ? ? ? ? ?



Constrained Matrices *

Syntax	Matrix Quantity	Dimensions
%On	Observed covariance matrix	NI x NI
%En	Expected covariance matrix	NI x NI
%Mn	Expected mean vector	1 x NI
%Pn	Expected proportions	NR x NC
%Fn	Function value	1 x 1

* to special quantities in previous groups



Matrix Algebra / Model

- Begin Algebra;

- $B = A * A'$;

- $C = B + B$;

- ...

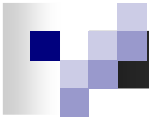
- End Algebra;

- Means [continuous] / Thresholds [categorical] X;

- Covariances X;

- Weight / Frequency X;

X: matrix or matrix formula



Unary Matrix Operations

Symbol	Name	Function	Example	Priority
\sim	Inverse	Inversion	$A\sim$	1
$`$	Transpose	Transposition	$A`$	1



Binary Matrix Operations

Symbol	Name	Function	Example	Priority
\wedge	Power	Element powering	A^B	2
*	Star	Multiplication	$A*B$	3
.	Dot	Dot multiplication	$A.B$	3
@	Kronecker	Kronecker product	$A@B$	3
&	Quadratic	Quadratic product	$A&B$	3
%	Eldiv	Element division	$A\%B$	3
+	Plus	Addition	$A+B$	4
-	Minus	Subtraction	$A-B$	4
	Bar	Horizontal adhesion	$A B$	4
_	Underscore	Vertical adhesion	A_B	4

Matrix Operations Priorities

(Mx manual p.59)

Symbol	Name	Function	Example	Priority
~	Inverse	Inversion	A~	1
`	Transpose	Transposition	A`	1
^	Power	Element powering	A^B	2
*	Star	Multiplication	A*B	3
.	Dot	Dot multiplication	A.B	3
@	Kronecker	Kronecker product	A@B	3
&	Quadratic	Quadratic product	A&B	3
%	Eldiv	Element division	A%B	3
+	Plus	Addition	A+B	4
-	Minus	Subtraction	A-B	4
	Bar	Horizontal adhesion	A B	4
_	Underscore	Vertical adhesion	A_B	4



Matrix Functions (Mx p. 64)

Keyword	Function	Restrictions	Dimensions
<code>\tr()</code>	Trace	$r=c$	1 x 1
<code>\det()</code>	Determinant	$r=c$	1 x 1
<code>\sum()</code>	Sum	None	1 x 1
<code>\prod()</code>	Product	None	1 x 1
<code>\max()</code>	Maximum	None	1 x 1
<code>\min()</code>	Minimum	None	1 x 1
<code>\abs()</code>	Absolute value	None	$r \times c$
<code>\exp()</code>	Exponent	None	$r \times c$
<code>\ln()</code>	Natural logarithm	None	$r \times c$
<code>\sqrt{}</code>	Square root	None	$r \times c$



Matrix Functions II

Keyword	Function	Restrictions	Dimensions
<code>\stand()</code>	Standardize	$r=c$	$r \times c$
<code>\mean()</code>	Mean of columns	None	$1 \times c$
<code>\cov()</code>	Covariance of cols	None	$c \times c$
<code>\pdfnor()</code>	Mv normal density	$r=c+2$	1×1
<code>\mnor()</code>	Mv normal integral	$r=c+3$	1×1
<code>\pchisq()</code>	Probability of Chi ²	$r=1 \ c=2$	1×2
<code>\d2v()</code>	Diagonal to vector	None	$\text{Min}(r,c) \times 1$
<code>\m2v()</code>	Matrix to vector	None	$rc \times 1$
<code>\part()</code>	Extract part of vector	None	variable



Specify Numbers/ Parameters

■ Numbers

- Matrix <name> <number list>
- Start/Value <name> <value> <element list>

■ Parameters

- Fix/Free <value> <element list>
- Equate <name GRC> <name GRC>
- Specify <name> <integer list>
- Bound low high <parameter list/element list>

■ Label Matrices

- Label Row/Column <name> <label list>



Options

■ Statistical Output

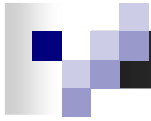
- Suppressing output: No_Output
- Appearance: NDecimals=n
- Residuals: RSiduals
- Adjusting Degrees of Freedom: DFreedom=n

■ Power Calculations

- Power = alpha,df

■ Confidence Intervals

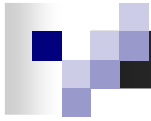
- Interval {@value} <matrix element list>



Options

■ Optimization options

- Bootstrap Estimates
- Randomizing Starting Values: THard=n
- Automatic Cold Restart: THard= n
- Jiggling Parameter Starting Values: Jiggle
- Confidence Intervals on Fit Statistics
- Comparative Fit Indices: Null
- Likelihood Ratio Statistics of Submodels: Issat/ Sat
- Check Identification of Model: Check



Fitting Submodels

- Multiple Fit
 - Option Multiple: Matrix/ Value/ Start/ Equate/ Fix/ Free/ Options
- Drop {@value} <parlist> <element list>
- Binary Save/Get <filename>
- Writing Matrices to Files
 - MXn = <filename>
- Writing Individual Likelihood Stats to Files:
 - MXnP = <filename>



Mx

- Graphical Interface
- Language

- www.vcu.edu/mx