Phenotypic Multi-group Analysis

Measurement Invariance
Continuous and categorical data

Gitta Lubke [1]   Irene Rebollo[2]

[1]Quantitative Psychology
University of Notre Dame

[2]Biological Psychology
Free University Amsterdam

Twin Workshop, Boulder 2008

Outline

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The general setting

- collected data are scores on items (e.g., questionnaire)
- questionnaire designed to measure a psychiatric disorder
  - or something else
- the grouping variable ‘gender’ defines two groups
  - or some other variable defines some other small number of groups
- interest in investigating differences between the two groups
- genetic decomposition can be added

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Observed items and theoretical constructs

- questionnaire items designed to measure a theoretical construct
- scores on the questionnaire items are observed
- constructs are usually not observable (e.g., IQ)
  - small number of theoretical construct
  - larger number observed variables
- goal is to compare groups with respect to the constructs

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- items that are designed to measure the same construct should have common variance
- common variance is captured by a latent variable
  - items measuring different aspects of math achievement
  - common variance of math items captured by (latent) math achievement
- latent variables (LV's) $\approx$ theoretical constructs
- questionnaire items = observed variables (OV's)
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Types of latent variable models

- different types of latent variable models
  - confirmatory factor analysis model (CFA)
  - item response models (IRT)
- CFA and IRT models are related
- both model types can be used for multiple group analyses
- goal is to compare groups with respect to a few latent variables
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Challenge

- prerequisite of group comparisons: show that items are measurement invariant
  - measurement invariance (MI) is absence of bias
- groups can be compared with respect to the latent variables only if MI holds
- to understand why biased items are problematic when comparing groups with respect to latent variables this section will cover
  - an introduction to the multi-group CFA model
  - conceptual approach to bias
  - theory of measurement invariance (MI)
  - how MI relates to the CFA model
  - how MI can be tested using the CFA model
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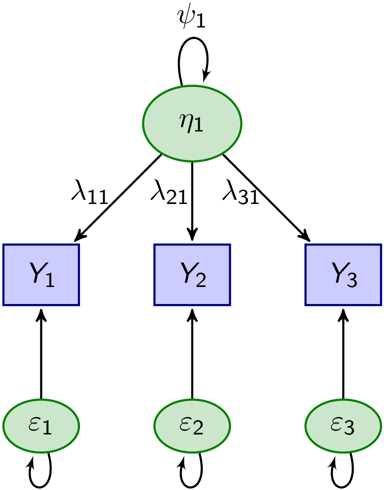
The common factor model: Path diagram



The common factor model: Model notation

- the model consists of *random variables* (=differ across subjects) and *model parameters* (=are the same across $i = 1, 2, \ldots, N$ subjects)
- the random variables for subject i are denoted as follows
  - $j = 1, 2, \ldots, J$ observed variables as Y_{ij}
  - $k = 1, 2, \ldots, K$ latent variables (='factors') as η_{ik}
  - measurement error of observed variable Y_j as ε_{ij}
- the model parameters are denoted as
  - regression intercept of Y_j as ν_j
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- the CFA model is a linear regression model, for subject i we have

$$Y_{ij} = \nu_j + \sum_{k=1}^{K} \lambda_{jk}\eta_{ik} + \varepsilon_{ij}$$

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Constraints to identify parameters

- latent variables have no scale
    - means? variance?
- latent variables η have to be scaled
- set means to zero
- either fix variance to 1 *or* one loading per factor to 1
    - since we are extending to multiple groups and may want to investigate variance differences between groups, we fix one loading to 1

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Practical 1: Single group phenotypic CFA model

Irene...

Extension to multiple groups

- the model can be fitted simultaneously in multiple groups
- the model is fitted to the means and covariances of the observed variables in each group g

$$E(\mathbf{Y_g}) = \nu_g + \Lambda_g \alpha_g$$
$$Cov(\mathbf{Y_g}) = \Lambda_g \Psi_g \Lambda_g' + \Theta_g$$

  - α_g: vector containing the factor means
  - Ψ_g: covariance matrix of the factors
  - Θ_g: the covariance matrix of the errors
- some constraints are needed to identify the model parameters, factor means and covariances, and error variances

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Constraining the model

- constraints to identify parameters
  - scale the latent variables in both groups
  - identify the mean model
- the resulting model permits groups to differ with respect to most model parameters
  - intercepts, loadings, error variances, factor means and variances
- question is whether we can compare groups with respect to the latent variables if all these parameters differ across groups

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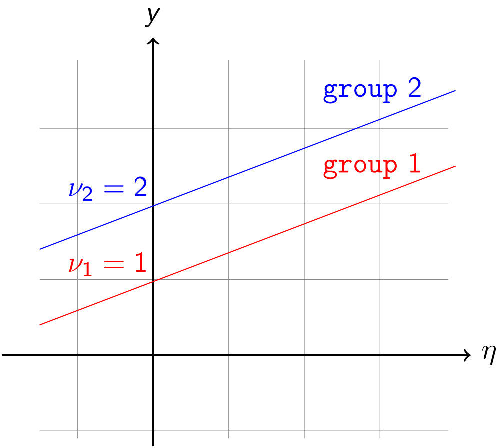
Practical 2: Two-group phenotypic CFA model

Irene...

Illustration: Group-specific intercepts



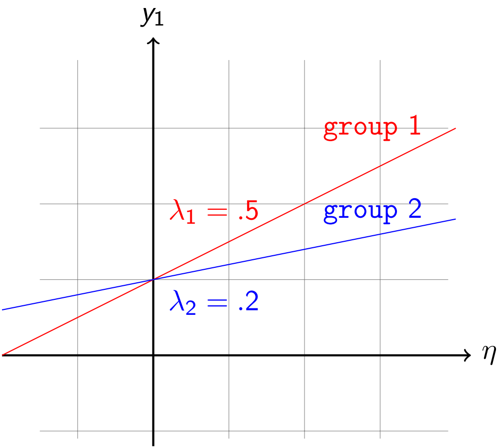
The groups differ with respect to the intercept ν.

$$E(Y_1) = 1 + \lambda\alpha$$
$$E(Y_2) = 2 + \lambda\alpha$$

Given the same score on η group 2 scores on average 1 unit higher on Y.

Consequence: We can't use Y to compare the groups with respect to η if the groups have different intercepts ν.

Illustration: Group-specific loadings



The groups differ with respect to the loading λ.

$$E(Y_1) = \nu + .5\alpha$$
$$E(Y_2) = \nu + .2\alpha$$

Given the same score on η group 2 scores on average increasingly higher (for positive η.

Consequence: We can't use Y to compare the groups with respect to η if the groups have different loadings λ.

Group-specific residual variances

- residual variance is the variability of the observed score Y at a given level of the latent variable η
- if groups differ, then one group has more variability on Y (i.e., more extreme scores) at each given level of η
- if we use Y to estimate η, the precision in that group is less
  - say we use Y to select subjects above a given level of η, then we make more mistakes in that group compared to the group with less residual variance
- it is sometimes argued that differences in residual variance are less problematic than differences in intercepts and loadings when comparing groups with respect to η

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Group-specific factor means and factor (co)variances

- these are the differences we are interested in when comparing groups!
- recall, the latent variables represent the theoretical constructs
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Definition of Measurement Invariance

$$f(y|\eta, s) = f(y|\eta)$$

- the distribution of the observed variable Y given the latent variables η and a grouping variable s equals the distribution of the observed variable Y given the latent variables η alone
  - in other words, the distribution of Y conditional on the latent variables does not differ across groups

- since the definition concerns the distribution of Y conditional on η, the distribution of η may differ across groups

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Summary: MI and the CFA model

- the definition of MI requires the distribution of $Y|\eta$ to be the same for all groups
- Y is assumed to be normally distributed
  - therefore means and covariance matrix of Y are sufficient statistics
- we impose the CFA model on the means and covariance matrix of Y in each group

$$\begin{aligned} E(\mathbf{Y_g}) &= \nu_g + \mathbf{\Lambda}_g E(\alpha_g) \\ Cov(\mathbf{Y_g}) &= \mathbf{\Lambda}_g \mathbf{\Psi}_g \mathbf{\Lambda}_g' + \mathbf{\Theta}_g \end{aligned}$$

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  - intercepts ν, loadings $\mathbf{\Lambda}$, residual variances $\mathbf{\Theta}$
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Testing MI in the CFA model (1)

- three consecutive tests using model comparisons
  - the three tests correspond to testing invariance of loadings, intercepts, and residual variances
- starting with the most lenient multi-group model (M_1), a series of four increasingly restrictive models can be fitted (M_1-M_4)
  - each of the three tests compare model M_k to the more lenient model M_{k-1} using a likelihood ratio test
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Practical 3: Testing MI in the phenotypic CFA model

Irene...

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Categorical data

- so far we assumed that **Y** is normally distributed
- more common: categorical outcomes
  - binary outcomes: endorsing a symptom, checklists
  - Likert data
- two approaches to model categorical data
  - latent response variable (or Y^*) approach (LRV)
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- latent response variable (LRV) formulation
  - see Agresti, Categorical Data analysis
- assumes that the 'true' response to an item is normally distributed
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  - we observe a categorized version of the LRV
- threshold structure is imposed on the the LRV
  - denote the LRV as Y^*, then for $C = 1, \ldots, c$ response categories we have $c + 1$ thresholds τ, where $\tau_1 = -\infty$ and $\tau_{c+1} = \infty$ and

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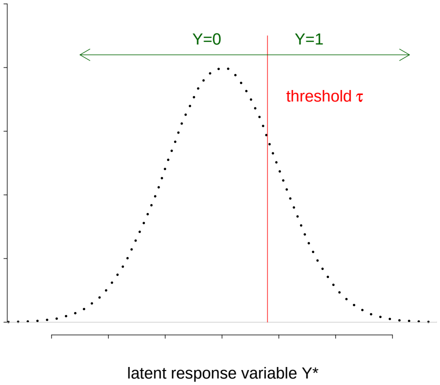
Y^*, τ, and categorical Y



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- the probability $P(Y=1) \equiv P(Y^* > \tau)$ is modeled directly
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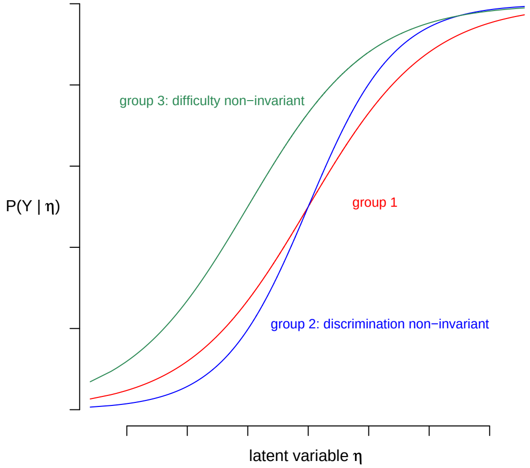
Group differences in $P(Y|\eta)$



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