Matrix Algebra
Mx and Likelihood

Michael C Neale PhD
Virginia Institute for Psychiatric and Behavioral Genetics VCU

International Workshop on Methodology for Genetic Studies

Boulder Colorado 3rd March 2008

Heuristic or Horrific?

- You already know a lot of it
- Economical and aesthetic
- Great for statistics

What you most likely know

- All about (1x1) matrices

Operation	Example	Result
Addition	2 + 2	
Subtraction	5 - 1	
Multiplication	2 x 2	
Division	12 / 3	

What you most likely know

- All about (1x1) matrices

Operation	Example	Result
Addition	2 + 2	4
Subtraction	5 – 1	4
Multiplication	2 x 2	4
Division	12 / 3	4

What you may guess

- Numbers can be organized in boxes, e.g.

1	2
3	4

What you may guess

- Numbers can be organized in boxes, e.g.

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

Matrix Notation

$$\underline{A}$$

Many Numbers

31 23 16 99 08 12 14 73 85 98 33 94 12 75 02 57 92 75 11
28 39 57 17 38 18 38 65 10 73 16 73 77 63 18 56 18 57 02
74 82 20 10 75 84 19 47 14 11 84 08 47 57 58 49 48 28 42
88 84 47 48 43 05 61 75 98 47 32 98 15 49 01 38 65 81 68
43 17 65 21 79 43 17 59 41 37 59 43 17 97 65 41 35 54 44
75 49 03 86 93 41 76 73 19 57 75 49 27 59 34 27 59 34 82
43 19 74 32 17 43 92 65 94 13 75 93 41 65 99 13 47 56 34
75 83 47 48 73 98 47 39 28 17 49 03 63 91 40 35 42 12 54
31 87 49 75 48 91 37 59 13 48 75 94 13 75 45 43 54 32 53
75 48 90 37 59 37 59 43 75 90 33 57 75 89 43 67 74 73 10
34 92 76 90 34 17 34 82 75 98 34 27 69 31 75 93 45 48 37
13 59 84 76 59 13 47 69 43 17 91 34 75 93 41 75 90 74 17
34 15 74 91 35 79 57 42 39 57 49 02 35 74 23 57 75 11 35

Matrix Notation

$$\underline{A}$$

Useful Subnotation

$$\underline{A}$$

2 2

Useful Subnotation

A
___

8 40

Matrix Operations

- Addition
- Subtraction
- Multiplication
- Inverse

Addition

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} & \\ & \end{bmatrix}$$

$$\underline{A} \quad + \quad \underline{B} \quad =$$

Addition

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 6 & 8 \\ 10 & 12 \end{bmatrix}$$

$$\underline{A} + \underline{B} = \underline{C}$$

Addition Conformability

To add two matrices A and B:

- # of rows in A = # of rows in B
- # of columns in A = # of columns in B

Subtraction

$$\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} & \\ & \end{bmatrix}$$

$$\underline{B} \quad - \quad \underline{A} \quad =$$

Subtraction

$$\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix}$$

$$\underline{B} - \underline{A} = \underline{C}$$

Subtraction Conformability

- To subtract two matrices A and B:
- # of rows in A = # of rows in B
- # of columns in A = # of columns in B

Multiplication Conformability

- Regular Multiplication
- To multiply two matrices A and B:
- # of columns in A = # of rows in B
- Multiply: A (m x n) by B (n by p)

Multiplication General Formula

$$C_{ij} = \sum_{k=1}^{n} A_{ik} \times B_{kj}$$

Multiplication I

$$\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} \times \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}$$

$$\underline{A} \times \underline{B} =$$

Multiplication II

$$\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} \times \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} (5x1) & \\ & \end{bmatrix}$$

$$\underline{A} \times \underline{B} = \underline{C}$$

$$C_{11} = \sum_{k=1}^{n} A_{11} \times B_{11}$$

Multiplication III

$$\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} \times \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} (5x1)+(6x3) & \\ & \end{bmatrix}$$

$$\underline{A} \times \underline{B} = \underline{C}$$

$$C_{11} = \sum_{k=2}^{n} A_{12} \times B_{21}$$

Multiplication IV

$$\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} \times \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 23 & (5 \times 2)+(6 \times 4) \\ & \end{bmatrix}$$

$$\underline{A} \times \underline{B} = \underline{C}$$

$$C_{12} = \sum_{k=1}^{n} A_{1k} \times B_{k2}$$

Multiplication V

$$\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} \times \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 23 & 34 \\ (7 \times 1)+(8 \times 3) & \end{bmatrix}$$

$$\underline{A} \times \underline{B} = \underline{C}$$

$$C_{21} = \sum_{k=1}^{n} A_{2k} \times B_{k1}$$

Multiplication VI

$$\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} \times \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 23 & 34 \\ 31 & (7 \times 2)+(8 \times 4) \end{bmatrix}$$

$$\underline{A} \times \underline{B} = \underline{C}$$

$$C_{22} = \sum_{k=1}^{n} A_{2k} \times B_{k2}$$

Multiplication VII

$$\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} \times \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 23 & 34 \\ 31 & 46 \end{bmatrix}$$

$$\underline{A} \times \underline{B} = \underline{C}$$

$m \times n$ $n \times p$ $m \times p$

Transpose

- Usually denoted by ’
- Sometimes ^{T}

- Exchanges rows and columns
- (m x n) matrix becomes (n x m)
- $A_{ij} = A_{ji}$

Inner Product of a Vector

- (Column) Vector **c** (n x 1)

$$\underline{\mathbf{c}} = \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 4 & 1 \end{bmatrix} \times \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} (2x2)+(4x4)+(1x1) \end{bmatrix} = \begin{bmatrix} 21 \end{bmatrix}$$

$$\underline{\mathbf{c}}' \qquad \underline{\mathbf{c}} \qquad = \qquad \underline{\mathbf{c}}'\,\underline{\mathbf{c}}$$

Outer Product of a Vector

- (Column) vector **c** (n x 1) $\underline{\mathbf{c}} = \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix}$

$$\begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix} \times \begin{bmatrix} 2 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 8 & 2 \\ 8 & 16 & 4 \\ 2 & 4 & 1 \end{bmatrix}$$

$$\underline{\mathbf{c}} \qquad\qquad \underline{\mathbf{c}'} \qquad\qquad \underline{\mathbf{c}}\,\underline{\mathbf{c}'}$$

Inverse

- A number can be divided by another number - How do you divide matrices?
- Note that a / b = a x 1 / b
- And that a x 1 / a = 1
- 1 / a is the inverse of a

Unary operations: Inverse

- Matrix ‘equivalent’ of 1 is the identity matrix

$$\underline{\mathbf{I}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

- A*I = I*A = A

- Find A^{-1} such that $A^{-1} * A = I$

Unary Operations: Inverse

- Inverse of (2 x 2) matrix
  - Find determinant
  - Swap a_{11} and a_{22}
  - Change signs of a_{12} and a_{21}
  - Divide each element by determinant
  - Check by pre- or post-multiplying by inverse

Inverse of 2 x 2 matrix

- Find the determinant

  $= (a_{11} \times a_{22}) = (a_{21} \times a_{12})$

  For

$$\underline{A} = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$$

det(**A**) = (2x3) – (1x5) = 1

Inverse of 2 x 2 matrix

- Swap elements a_{11} and a_{22}

  Thus

$$\underline{A} = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$$

  becomes

$$\begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$$

Inverse of 2 x 2 matrix

- Change sign of a_{12} and a_{21}

  Thus

$$\underline{A} = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$$

  becomes

$$\begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$$

Inverse of 2 x 2 matrix

- Divide every element by the determinant

  Thus

$$\underline{A} = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$$

becomes

$$\begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$$

(luckily the determinant was 1)

Inverse of 2 x 2 matrix

- Check results with $\mathbf{A}^{-1}\,\mathbf{A} = \mathbf{I}$

  Thus

$$\begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix} \times \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$$

  equals

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Intro to Mx Script Language

General Comments

- Case insensitive, except for filenames under Unix
- Comments
  - Anything following a !
  - Sections delimited by lines beginning /* and ending */
- Blank lines are ignored
- Keywords: identified by first 2 letters, BUT recommended to use full words

Job Structure

- Three types of groups:
  - Data, Calculation, Constraint
- Number of groups indicated by
  - #NGroups 3
  - at the beginning of job
- Jobs can be stacked in one run

Group Structure

- Title
- Group type: data, calculation, constraint
  - [Read observed data, Select, Labels]
- Matrices declaration
  - [Specify numbers, parameters, etc.]
- Algebra section and/or Model statement
  - [Options]
- End

A pathetic Mx script example

```
#ngroups 1
Title
Calculation
Begin Matrices;
End Matrices;
End
```

A less pathetic Mx script example

```
#ngroups 1
Title
Calculation
Begin Matrices;
 A full 2 2
End Matrices;
 Matrix A
 2 5
 1 3
Begin Algebra;
 B = A~ ;
End Algebra;
End
```

Read Observed Data

- Data NInputvars=2 [NObservations=123]
- CMatrix/ Means/ CTable/
  - summary statistics
  - read from script / file (File=filename)
- Rectangular/ Ordinal / VLength
  - raw data
  - read from script / file (File=filename)
- Select variables ; [by number/label]
- Labels variables

Matrix Declaration

- Group 1
- Begin Matrices;
  - C Full 2 3 Free   ! [name type rows columns free]
  - ! more matrices   ! default element is fixed at 0
- End Matrices;

- Group 2
- Begin Matrices = Group 1;
  - ! copies all matrices from group 1
  - D Full 2 3 = C1   ! equates D to C of group 1

Matrix Types (Mx manual p.56)

Type	Structure	Shape	Modifiable
Zero	Null (zeros)	Any	0
Unit	Unit (ones)	Any	0
Iden	Identity	Square	0
Diag	Diagonal	Square	r
SDiag	Subdiagonal	Square	r(r-1)/2
Stand	Standardized	Square	r(r-1)/2
Symm	Symmetric	Square	r(r+1)/2
Lower	Lower triangular	Square	r(r+1)/2
Full	Full	Any	r x c
Computed	Equated to	Any	0

Matrices

Example Command	Specification Matrix	Values
A Zero 2 3 Free	0 0 0 0 0 0	0 0 0 0 0 0
B Unit 2 3 Free	0 0 0 0 0 0	1 1 1 1 1 1
C Iden 3 3 Free	0 0 0 0 0 0 0 0 0	1 0 0 0 1 0 0 0 1
D Izero 2 5 Free	0 0 0 0 0 0 0 0 0 0	1 0 0 0 0 0 1 0 0 0
E Ziden 2 5 Free	0 0 0 0 0 0 0 0 0 0	0 0 0 1 0 0 0 0 0 1

Matrices II

Example Command	Specification Matrix	Values
F Diag 3 3 Free	1 0 0 0 2 0 0 0 3	? 0 0 0 ? 0 0 0 ?
G SDiag 3 3 Free	0 0 0 1 0 0 2 3 0	0 0 0 ? 0 0 ? ? 0
H Stand 3 3 Free	0 1 2 1 0 3 2 3 0	1 ? ? ? 1 ? ? ? 1

Matrices III

Example Command	Specification Matrix	Values
I Symm 3 3 Free	1 2 4 2 3 5 4 5 6	? ? ? ? ? ? ? ? ?
J Lower 3 3 Free	1 0 0 2 3 0 4 5 6	? 0 0 ? ? 0 ? ? ?
K Full 2 4 Free	1 2 3 4 5 6 7 8	? ? ? ? ? ? ? ?

Matrix Algebra / Model

- Begin Algebra;
  - B = A*A';
  - C = B+B;
  - ...
- End Algebra;

- Means [continuous] / Thresholds [categorical] X;
- Covariances X;
- Weight / Frequency X;

X: matrix or matrix formula

Unary Matrix Operations

Symbol	Name	Function	Example	Priority
~	Inverse	Inversion	A~	1
`	Transpose	Transposition	A`	1

Binary Matrix Operations

Symbol	Name	Function	Example	Priority
^	Power	Element powering	A^B	2
*	Star	Multiplication	A*B	3
.	Dot	Dot multiplication	A.B	3
@	Kronecker	Kronecker product	A@B	3
&	Quadratic	Quadratic product	A&B	3
%	Eldiv	Element division	A%B	3
+	Plus	Addition	A+B	4
-	Minus	Subtraction	A-B	4
|	Bar	Horizontal adhesion	A|B	4

Matrix Operations Priorities
(Mx manual p.59)

<table>
<tr><th>Symbol</th><th>Name</th><th>Function</th><th>Example</th><th>Priority</th></tr>
<tr><td>~</td><td>Inverse</td><td>Inversion</td><td>A~</td><td>1</td></tr>
<tr><td>`</td><td>Transpose</td><td>Transposition</td><td>A`</td><td>1</td></tr>
<tr><td>^</td><td>Power</td><td>Element powering</td><td>A^B</td><td>2</td></tr>
<tr><td>*</td><td>Star</td><td>Multiplication</td><td>A*B</td><td>3</td></tr>
<tr><td>.</td><td>Dot</td><td>Dot multiplication</td><td>A.B</td><td>3</td></tr>
<tr><td>@</td><td>Kronecker</td><td>Kronecker product</td><td>A@B</td><td>3</td></tr>
<tr><td>&</td><td>Quadratic</td><td>Quadratic product</td><td>A&B</td><td>3</td></tr>
<tr><td>%</td><td>Eldiv</td><td>Element division</td><td>A%B</td><td>3</td></tr>
<tr><td>+</td><td>Plus</td><td>Addition</td><td>A+B</td><td>4</td></tr>
<tr><td>-</td><td>Minus</td><td>Subtraction</td><td>A-B</td><td>4</td></tr>
<tr><td>|</td><td>Bar</td><td>Horizontal adhesion</td><td>A|B</td><td>4</td></tr>
<tr><td>_</td><td>Underscore</td><td>Vertical adhesion</td><td>A_B</td><td>4</td></tr>
</table>

Matrix Functions (Mx p. 64)

Keyword	Function	Restrictions	Dimensions
\tr()	Trace	r=c	1 x 1
\det()	Determinant	r=c	1 x 1
\sum()	Sum	None	1 x 1
\prod()	Product	None	1 x 1
\max()	Maximum	None	1 x 1
\min()	Minimum	None	1 x 1
\abs()	Absolute value	None	r x c
\exp()	Exponent	None	r x c
\ln()	Natural logaritm	None	r x c
\sqrt()	Square root	None	r x c

Matrix Functions II

Keyword	Function	Restrictions	Dimensions
\stand()	Standardize	r=c	r x c
\mean()	Mean of columns	None	1 x c
\cov()	Covariance of cols	None	c x c
\pdfnor()	Mv normal density	r=c+2	1 x 1
\mnor()	Mv normal integral	r=c+3	1 x 1
\pchi()	Probability of Chi^2	r=1 c=2	1 x 2
\d2v()	Diagonal to vector	None	Min(r,c) x 1
\m2v()	Matrix to vector	None	rc x 1
\part()	Extract part of vector	None	variable

Specify Numbers/ Parameters

- Numbers
  - Matrix <name> <number list>
  - Start/Value <name> <value> <element list>
- Parameters
  - Fix/Free <value> <element list>
  - Equate <name GRC> <name GRC>
  - Specify <name> <integer list>
  - Bound low high <parameter list/element list>

Mx

- Graphical Interface
- Language

- www.vcu.edu/mx

Summarizing Variation and Likelihood

Overview

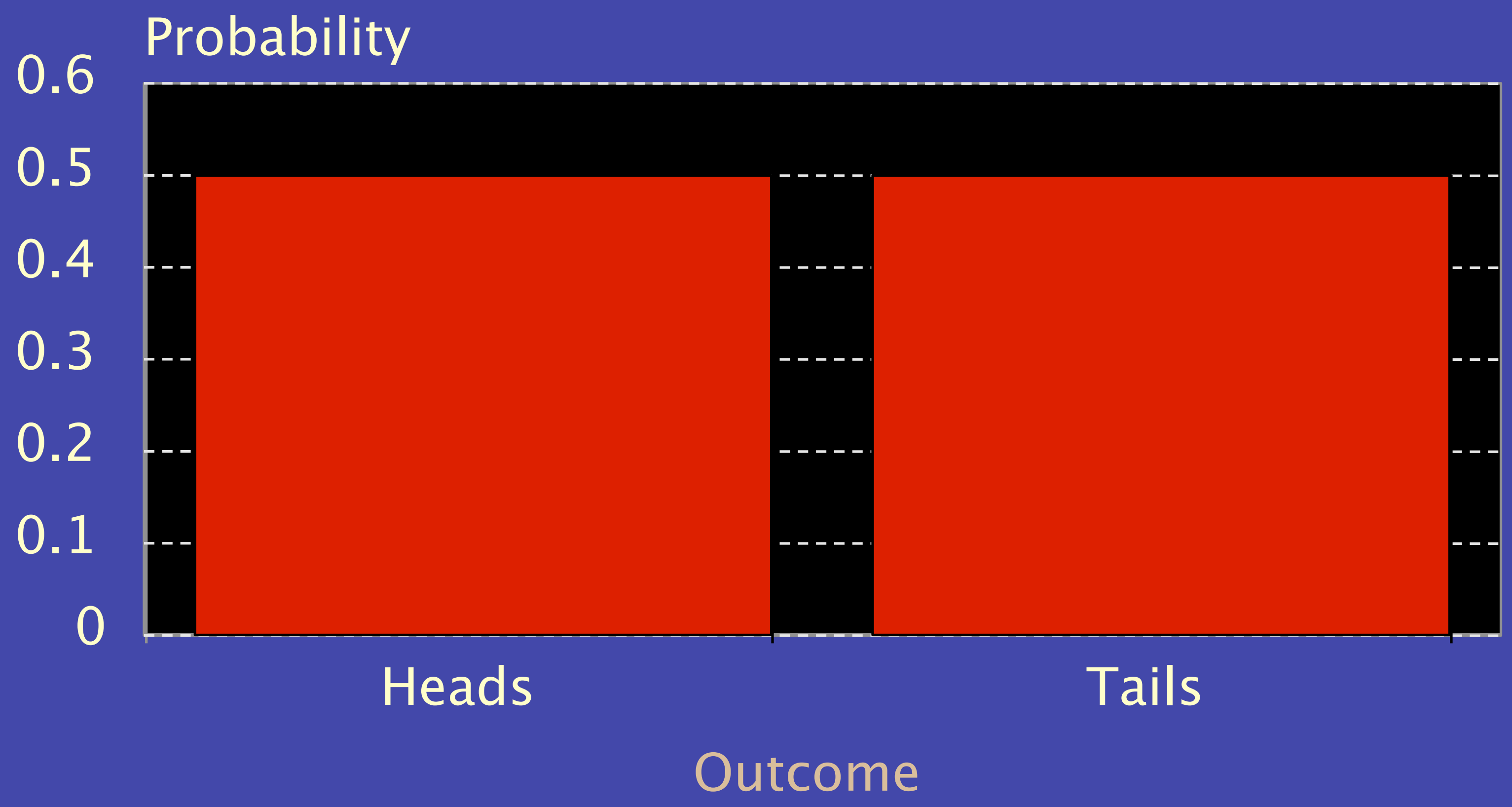
- Mean/Variance/Covariance
  - Calculating
  - Estimating by ML
- Matrix Algebra
- Normal Likelihood Theory
- Mx script language

Computing Mean

- Formula $\sum(x_i)/N$
- Can compute with
  - Pencil
  - Calculator
  - SAS
  - SPSS
  - Mx

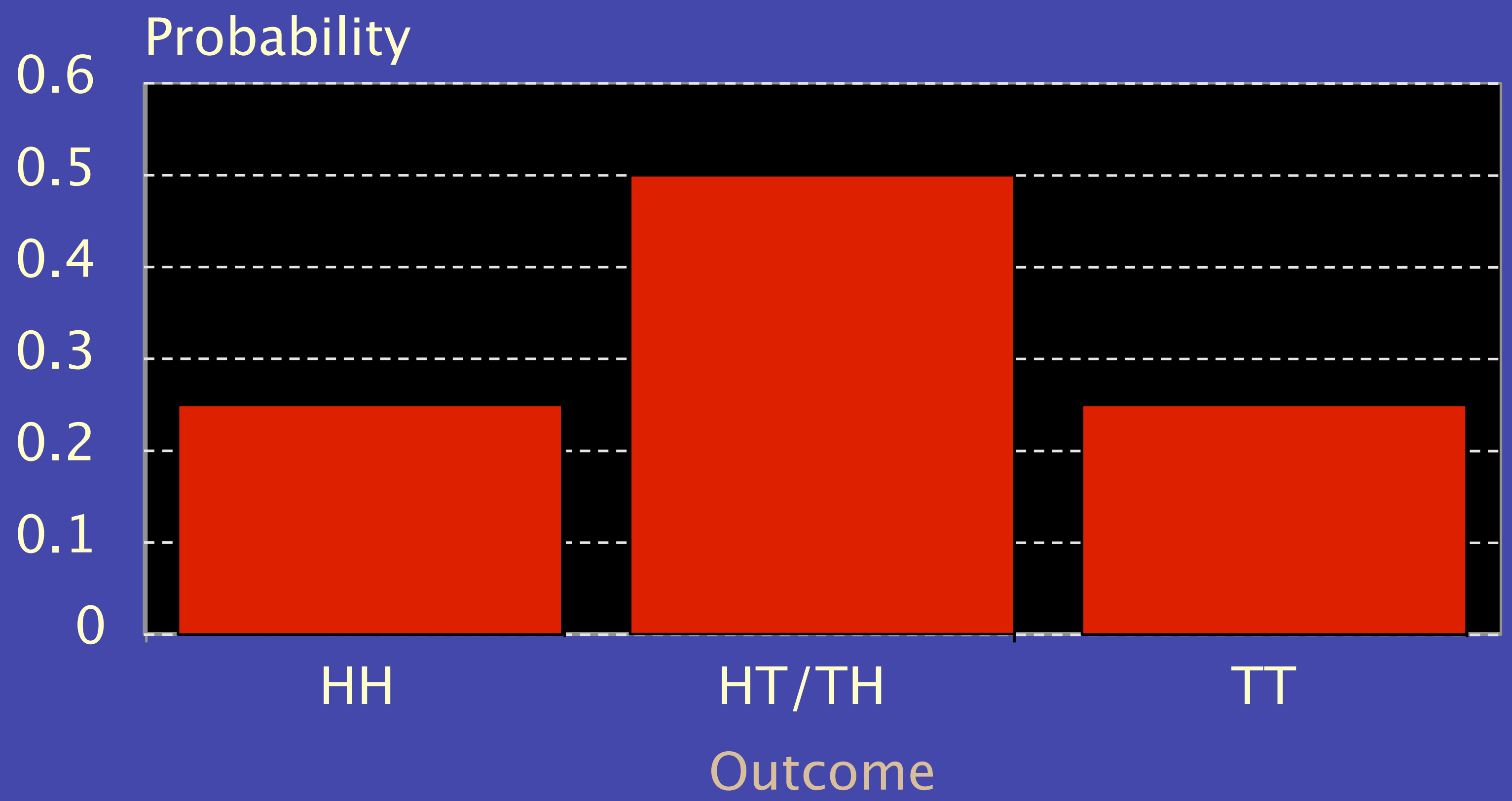
One Coin toss

2 outcomes



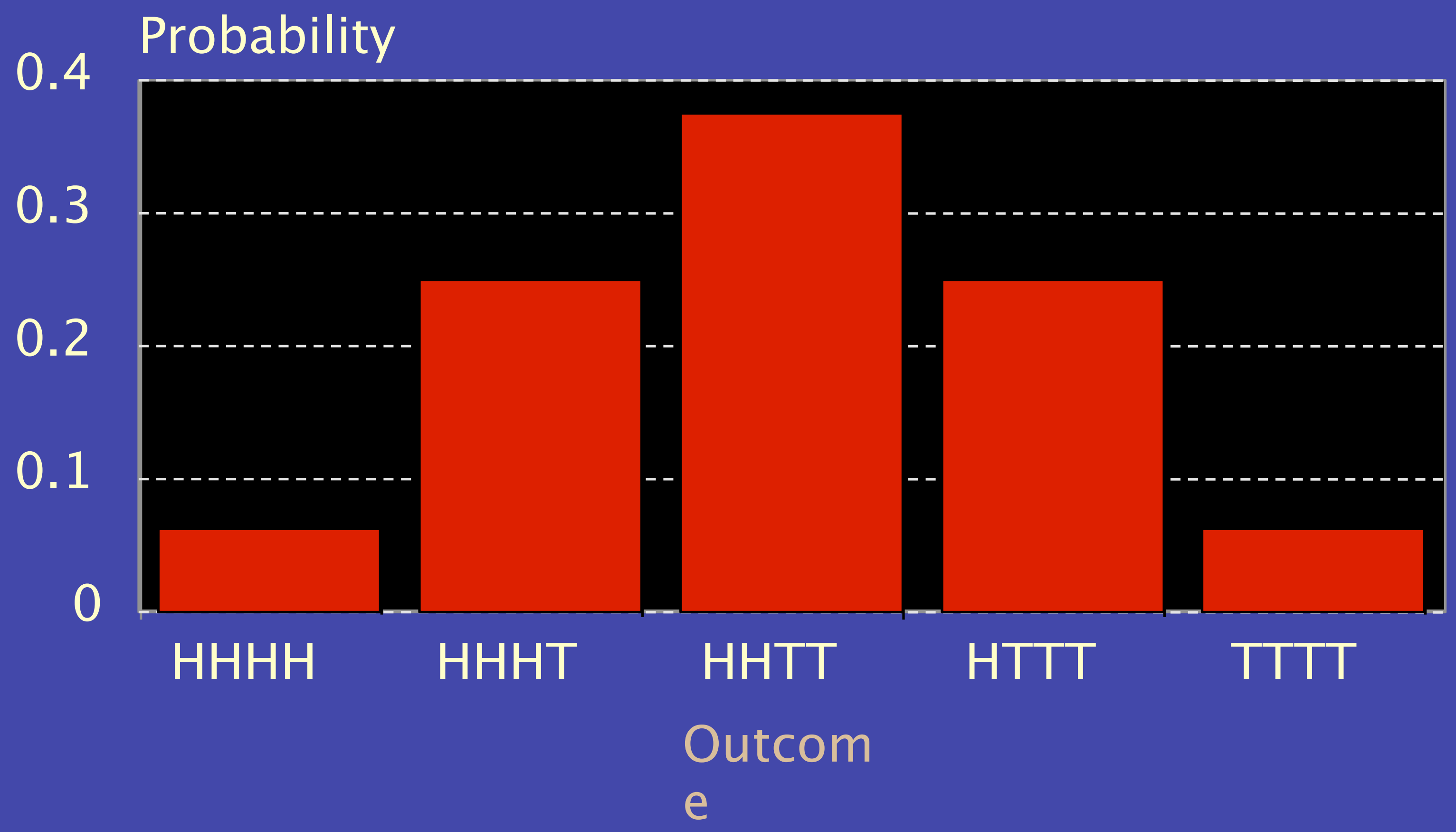
Two Coin toss

3 outcomes



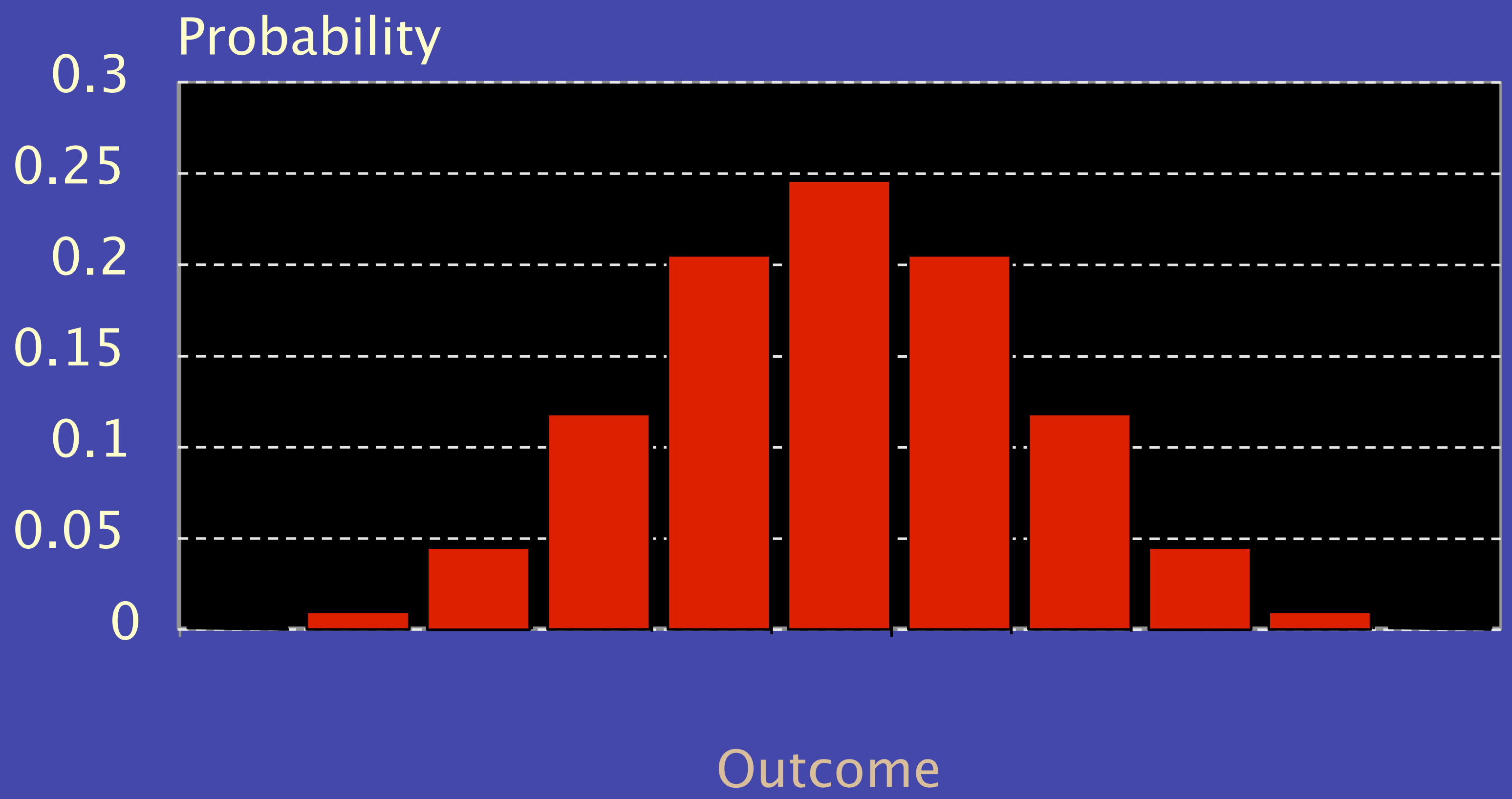
Four Coin toss

5 outcomes



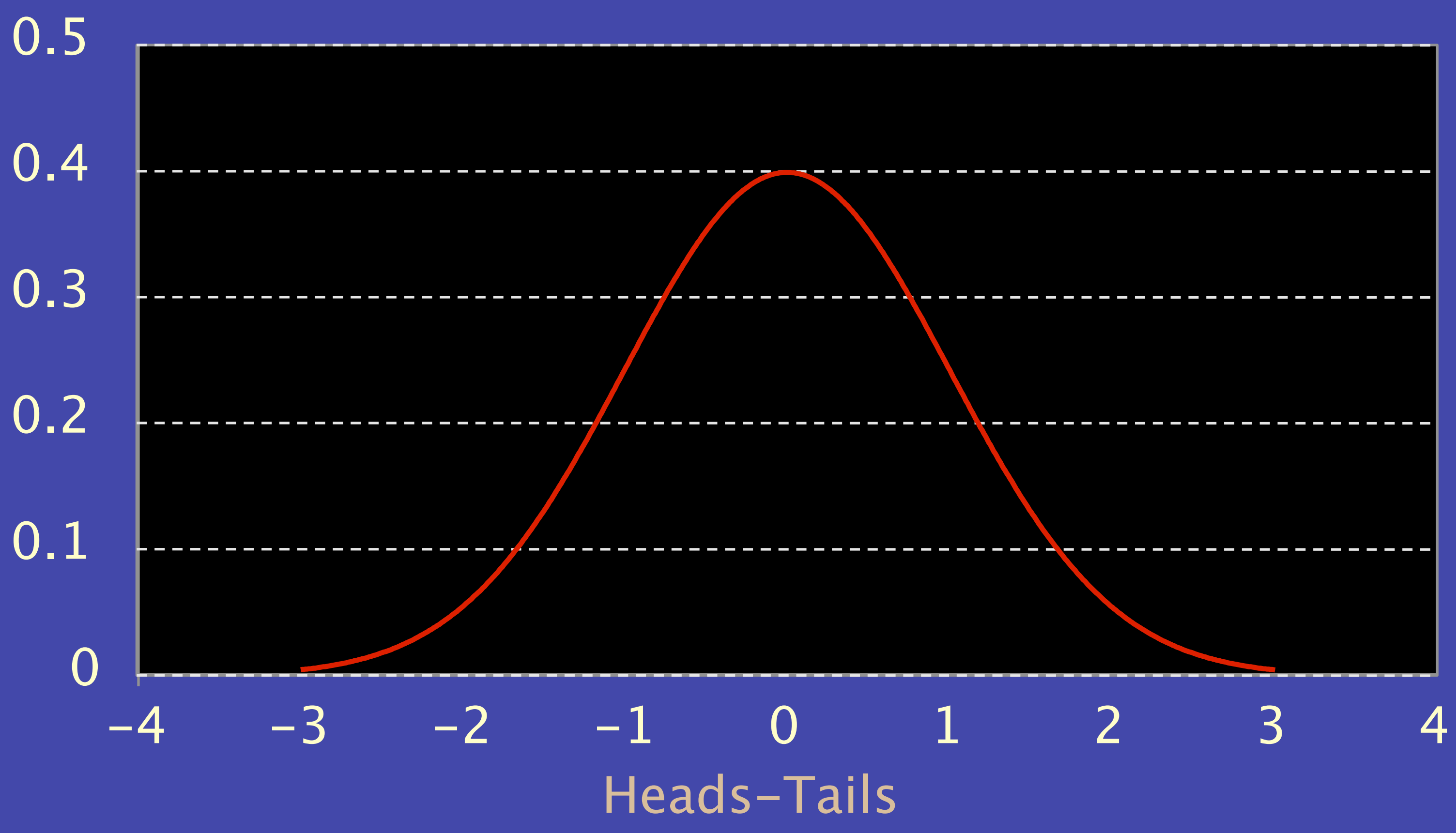
Ten Coin toss

9 outcomes



Fort Knox Toss

Infinite outcomes



De Moivre 1733 Gauss 1827

Dinosaur (of a) Joke

- Elk:
  The Theory by A. Elk brackets Miss brackets. My theory is along the following lines.

- Host:
  Oh God.

- Elk:
  All brontosauruses are thin at one end, much MUCH thicker in the middle, and then thin again at the far end.

Pascal's Triangle

Frequency	Probability
1	1/1
1 1	1/2
1 2 1	1/4
1 3 3 1	1/8
1 4 6 4 1	1/16
1 5 10 10 5 1	1/32
1 6 15 20 15 6 1	1/64
1 7 21 35 35 21 7 1	1/128

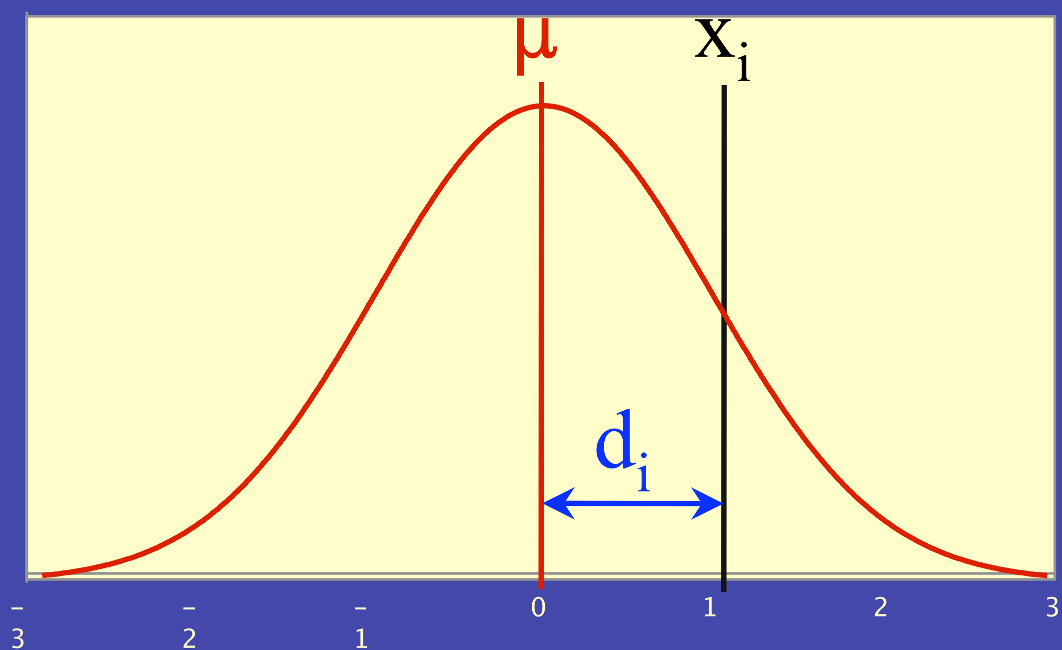
Pascal's friend Chevalier de Mere 1654; Huygens 1657; Cardan 1501-1576

Variance

- Measure of Spread
- Easily calculated
- Individual differences

Average squared deviation

Normal distribution



$$\text{Variance} = \Sigma\ d_i^{\,2}/N$$

Measuring Variation

Weighs & Means

- Absolute differences?
- Squared differences?
- Absolute cubed?
- Squared squared?

Measuring Variation

Weighs & Means

- Squared differences

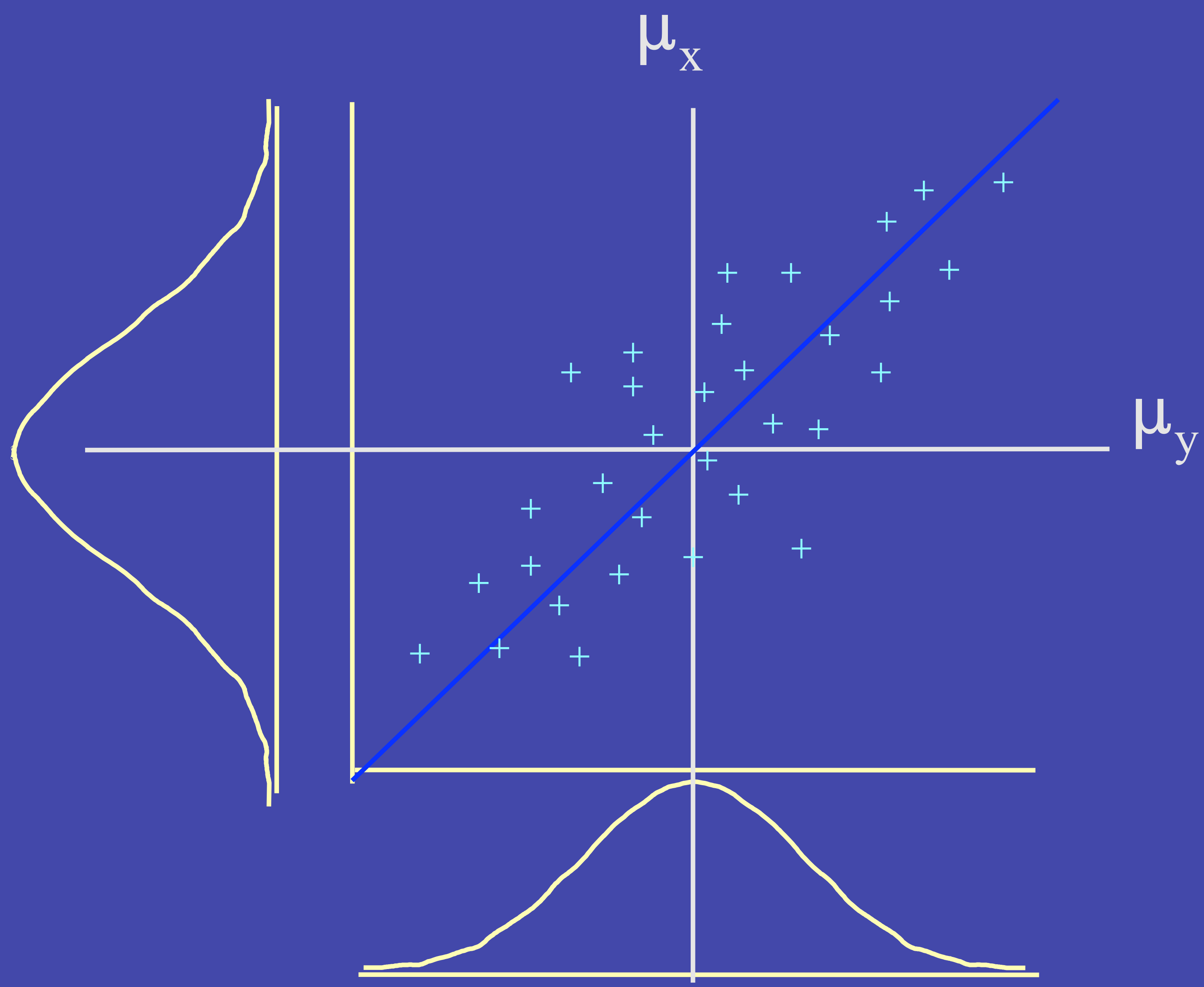
Fisher (1922) On the mathematical foundations of theoretical statistics. *Phil Trans Roy Soc London A* **222**:309-368

Squared has minimum variance under normal distribution

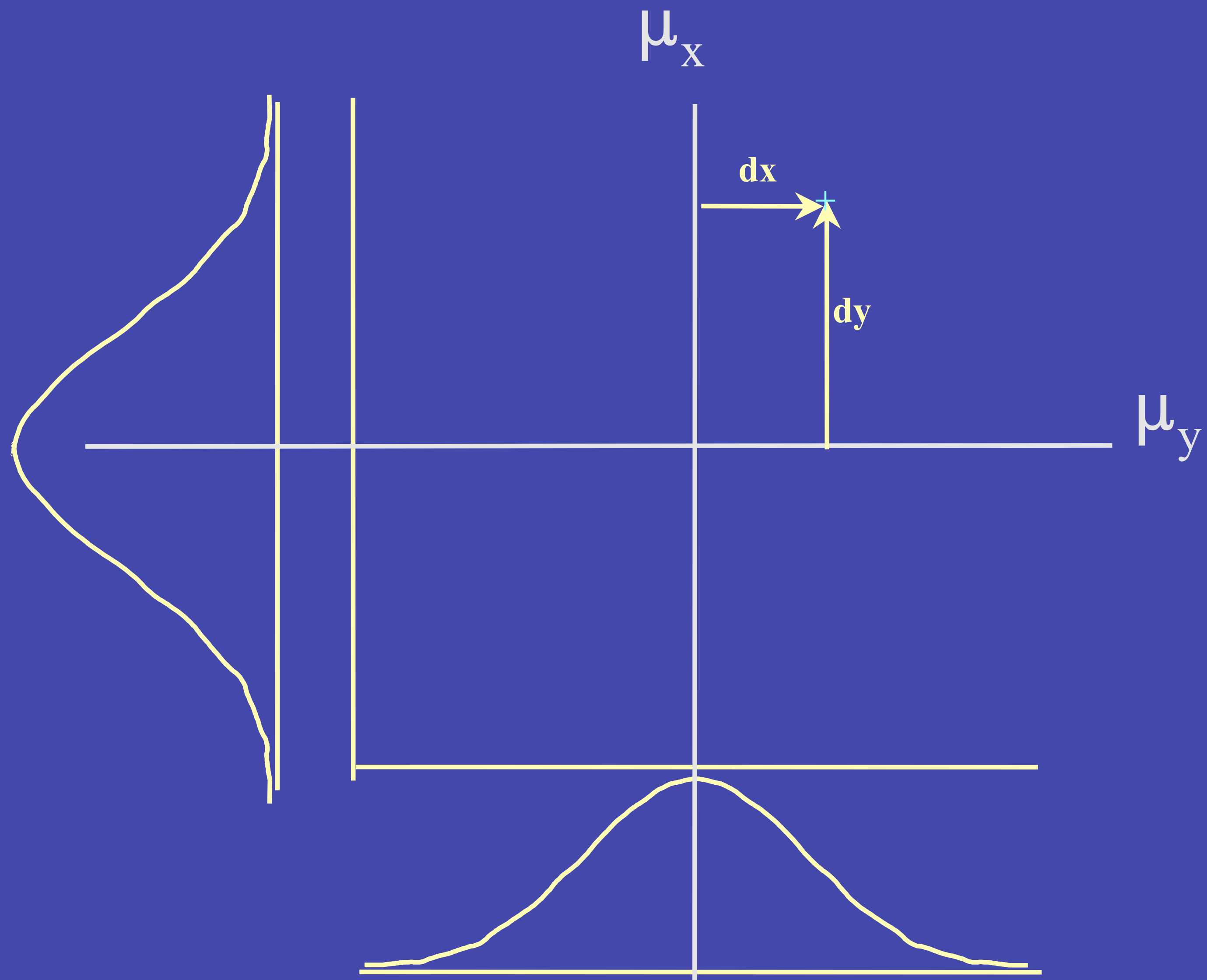
Covariance

- Measure of association between two variables
- Closely related to variance
- Useful to partition variance

Deviations in two dimensions



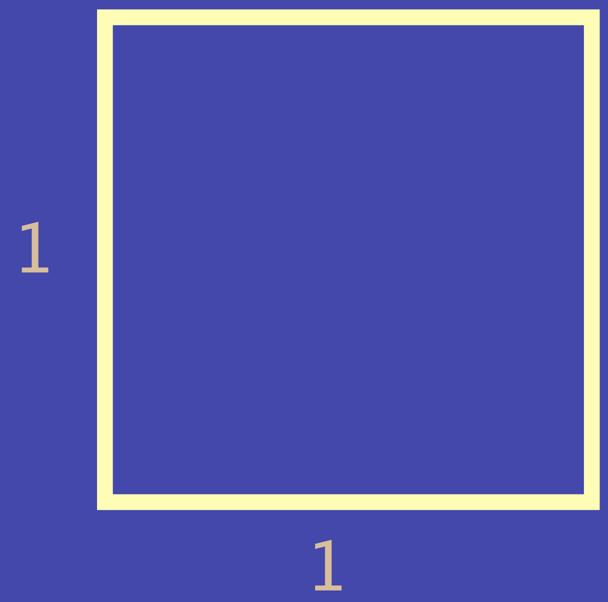
Deviations in two dimensions



Measuring Covariation

Concept: Area of a rectangle

- A square, perimeter 4
- Area 1



Measuring Covariation

Concept: Area of a rectangle

- A skinny rectangle, perimeter 4
- Area .25*1.75 = .4385

Measuring Covariation

Concept: Area of a rectangle

- Points can contribute negatively
- Area -.25*1.75 = -.4385

Measuring Covariation

Covariance Formula: Average cross-product of deviations from mean

$$\sigma_{xy} = \frac{\sum (x_i - \mu_x)(y_i - \mu_y)}{N}$$

Correlation

- Standardized covariance
- Lies between -1 and 1

$$r_{xy} = \frac{\sigma_{xy}}{\sqrt{\sigma_x^2 * \sigma_y^2}}$$

Summary

Formulae for sample statistics; i=1…N observations

$$\mu = (\Sigma x_i)/N$$

$$\sigma_x^2 = \Sigma \; (x_i - \mu_x)^2 \,/\, (N)$$

$$\sigma_{xy} = \Sigma (x_i - \mu_x)(y_i - \mu_y) \,/\, (N)$$

$$r_{xy} = \frac{\sigma_{xy}}{\sqrt{\sigma_x^2 \;\; \sigma_x^2}}$$

Variance covariance matrix

Several variables

Var(X)	Cov(X,Y)	Cov(X,Z)
Cov(X,Y)	Var(Y)	Cov(Y,Z)
Cov(X,Z)	Cov(Y,Z)	Var(Z)

Variance covariance matrix

Univariate Twin Data

$$\begin{bmatrix} Var(Twin1) & Cov(Twin1,Twin2) \\ Cov(Twin2,Twin1) & Var(Twin2) \end{bmatrix}$$

Only suitable for complete data
Good conceptual perspective

Conclusion

- Means and covariances
- Basic input statistics for “Traditional SEM”
- Easy to compute
- Can use raw data instead

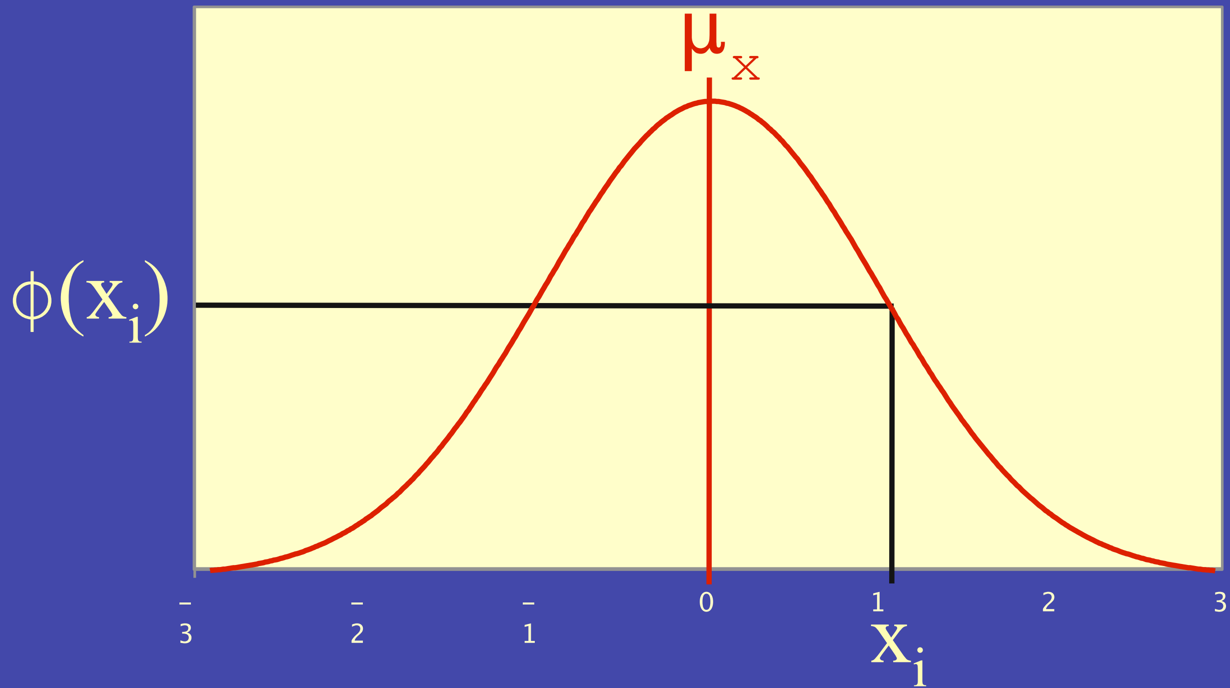
Likelihood computation

Calculate height of curve

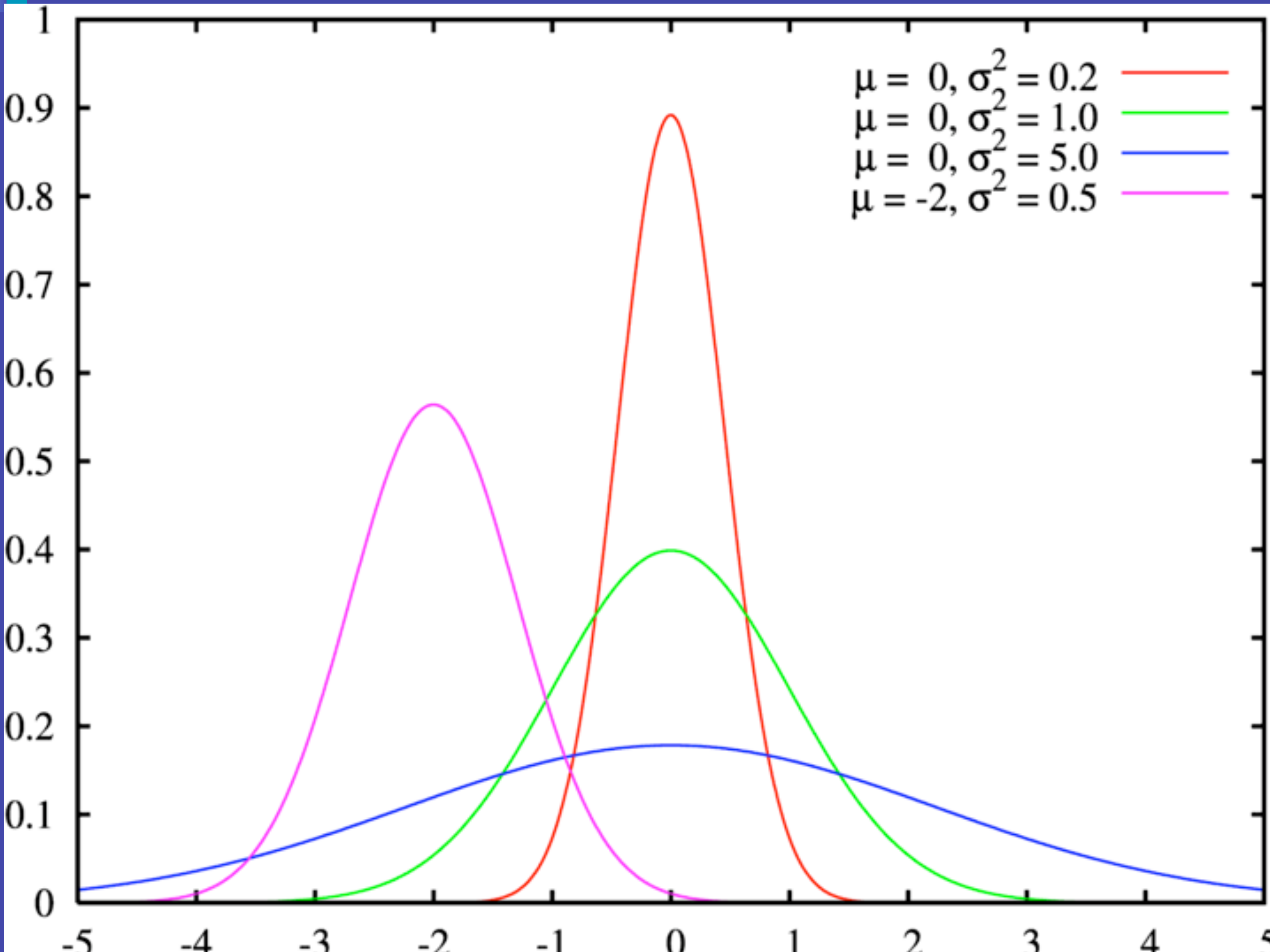
- Univariate - height of normal pdf
  - $\phi(x) =$
  - $(2\Pi\sigma^2)^{-.5}\ e^{-.5((x_i - \mu)^2)/\sigma^2)}$
- Multivariate - height of multinormal pdf
  - $|2\Pi\Sigma|^{-n/2}\ e^{-.5((\mathbf{x}_i - \mu)\Sigma^{-1}(\mathbf{x}_i - \mu)')}$

Height of normal curve

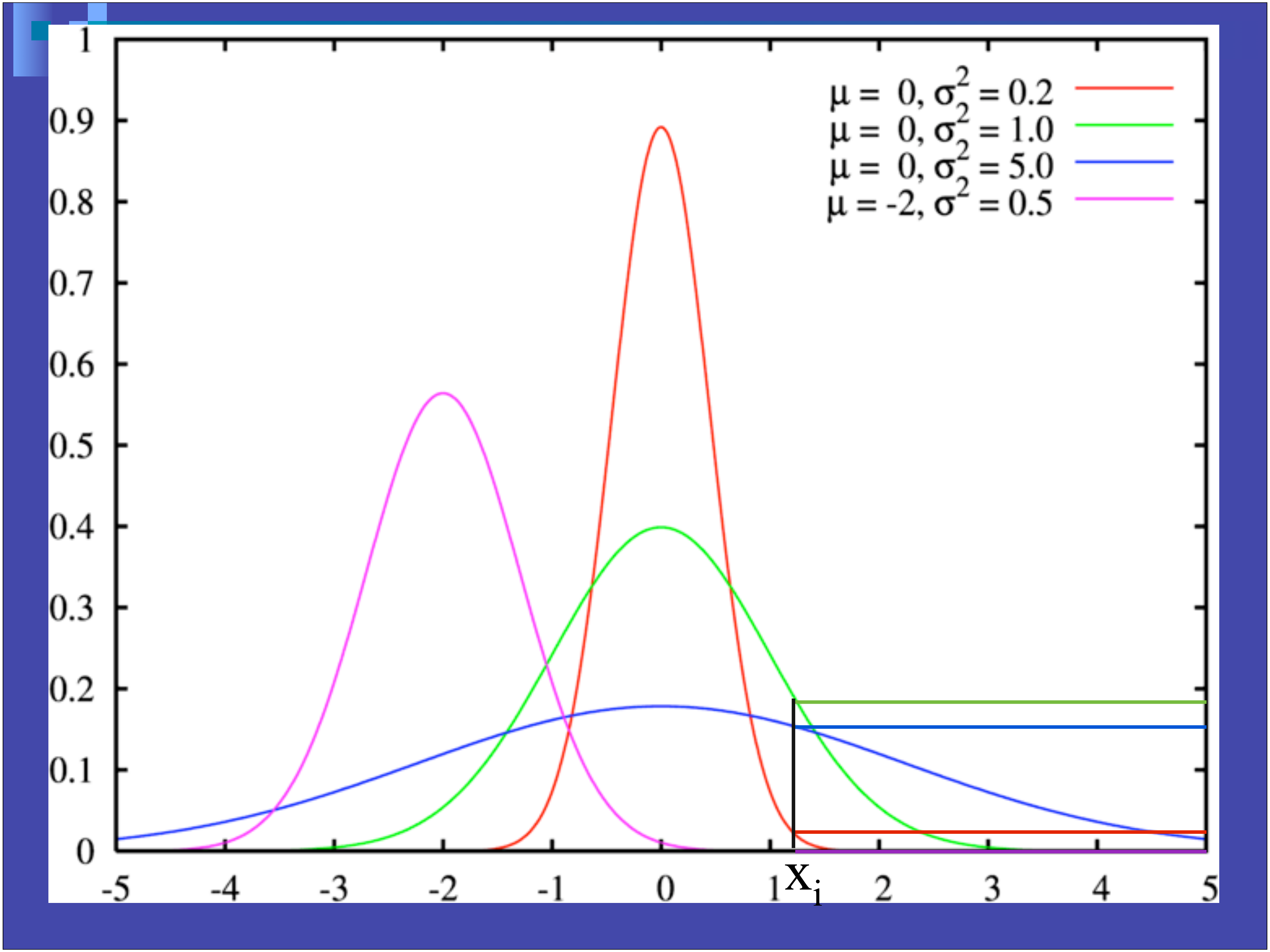
Probability density function



$\phi(x_i)$ is the likelihood of data point x_i for particular mean & variance estimates



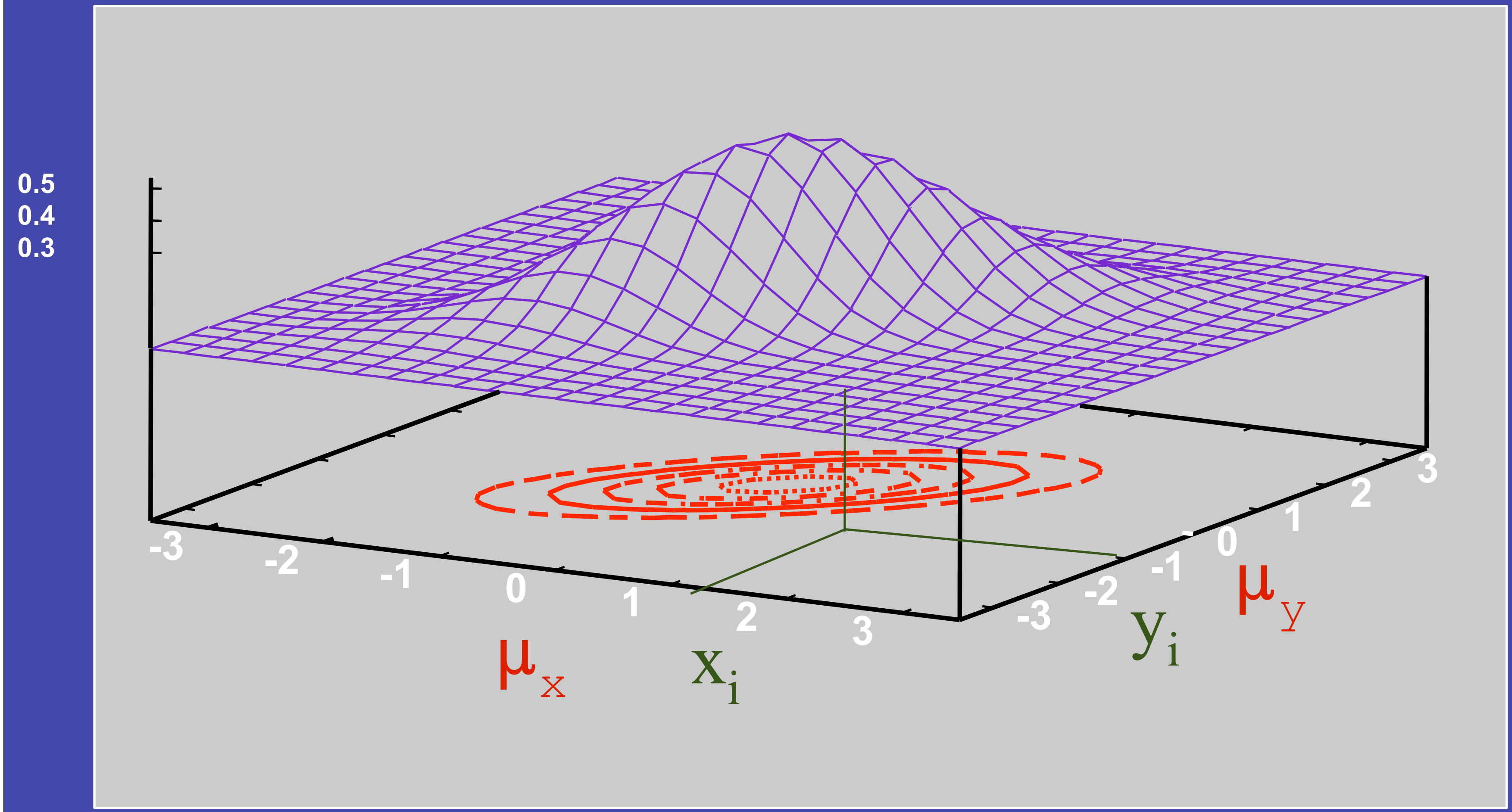
μ = 0, σ2 = 0.2
μ = 0, σ2 = 1.0
μ = 0, σ2 = 5.0
μ = -2, σ2 = 0.5
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
-5
-4
-3
-2
-1
0
1
2
3
4
5



$\mu = 0, \sigma^2 = 0.2$
$\mu = 0, \sigma^2 = 1.0$
$\mu = 0, \sigma^2 = 5.0$
$\mu = -2, \sigma^2 = 0.5$
X_i
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
-5 -4 -3 -2 -1 0 1 2 3 4 5

Height of bivariate normal curve

An unlikely pair of (x,y) values



Exercises: Compute Normal PDF

- Get used to Mx script language
- Use matrix algebra
- Taste of likelihood theory