Confirmation of Riemann's Zeta Function Conjecture

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(Dated: 25th Oct 2006 - draft v1)

This work is dedicated to the genius of Charles Musès [1].

Abstract: Utilizing Charles Musès' 2-Dimensional r- ε model of hyperbolic geometry, equations for boundary parallels are determined. We then show that zeroing a factor in the Zeta functions' Infinite Product of Primes Representation implies a statement about a hyperbolic plane in 4-Dimensional Hyperbolic Quaternion space. Solutions to Riemann's conjecture become obvious and a formula is provided to calculate solutions. Solutions with real part a = ½ hold, but it is the norm of the complex number s= a +bi that has more significance.

Introduction

The task at hand is to construct a novel solution to the long standing mathematical question conjectured by Riemann and relating to the zeros of the Zeta function with complex argument.

Riemann was keen to explore the following function and its so-called non-trivial zero solutions [2]:

$$\zeta(s) = \sum_{0}^{\infty} \frac{1}{n^s} = \prod_{p_j} \frac{1}{1 - \frac{1}{p_j^{\,s}}} \tag{1}$$

Here p_j is the j^th^ prime,

s = a + bi is an *i-complex number* such that a, b $\in$ R and $i^2 = -1$,

and the product is over the infinitude of primes.

Riemann's famous conjecture is that all the "interesting" zeros of this complex Zeta function lie on a line in the (r,i) plane with real part a = ½.

We shall discover that once the appropriate tools are introduced, this conjecture becomes a fairly straight forward question to answer. An understanding of Hyperbolic geometry is essential, and thanks to Charles Musès' [4] and Kevin Carmody [5,6,7], this task will be as simple as learning ordinary complex numbers in high school. Representation is everything! Furthermore, a rudimentary understanding of the 4-D Hyperbolic Quaternion geometry is essential. Hyperbolic Quaternions are hypernumbers of the form

$$s = a + bi_1 + c\varepsilon_2 + d\varepsilon_3 \qquad (a,b,c,d \in \mathrm{R}) \tag{2}$$

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and are a type of number composed of a *real unit,* a single i_1 *unit* and two copies of a *hyperbolic complex number unit,* denoted here as ε_2 *and* ε_3, *(where* $\varepsilon_n^{\;2} = 1$ and $\varepsilon_n \neq 1$). Again thanks to Musès [4] and Carmody [5,6,7] the task of understanding Hyperbolic Quaternions has been simplified.

In introducing hyperbolic geometry, the present work initially straddles between understandings developed from the axiomatic approach to hyperbolic geometry, as developed by Bolyai, Gauss, and Lobachevsky [3], and to the relatively more recent works of Musès and Carmody. The intention is to hasten the present proof while revealing some of the elegance of the *r-* ε *hyperbolic model.*

We start by constructing a very useful trigonometric expression for the general factor in the Infinite Product over Prime representation of the Zeta function.

1. *A Simple Trigonometric form of the general factor of the Zeta Function*

Referring to Equation (1), we define the factor function $F(p_j)$ as:

$$F(p_j) = \frac{1}{1 - \frac{1}{p_j^{\,s}}} \tag{3}$$

We shall approach the Riemann Zeta Conjecture by first identifying the conditions under which this function equals zero. The present work will assume that an infinite product with all factors non-zero, except one, will multiple to equal zero. (We are on risky ground right from the start! [8,9].)

Let us construct equation (3) from 'the ground up'. In the process we can cast the factor function $F(p_j)$ *into a more revealing trigonometric form.*

Using the definition of complex logarithms, and re-writing s in an equivalent form;

$$s = (-si)i \equiv s'i \tag{4}$$

and by utilizing the identity $e^{i\theta} = \cos\theta + i\sin\theta$, we expand p^s as:

$$p^s = e^{s'\ln p i} = \underset{i}{C}(s'\ln p) + i\,\underset{i}{S}(s'\ln p) \tag{5}$$

{Appendix A explains the use of the new nomenclature, $\underset{i}{C}$ and $\underset{i}{S}$, for our familiar Cosine and Sine functions respectively. Generalizations of the trig functions in other hypernumber geometries are also introduced.}

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Since we are working in ordinary *i-complex* space, we can invert this number with impunity as long as p^s is non-zero:

$$\frac{1}{p^s} = \underset{i}{C}(s'\ln p) - i\,\underset{i}{S}(s'\ln p) \tag{6}$$

The next few steps in our construction are straightforward, again with provisos for division by zero and using some simple trig identities for double angles:

$$1 - \frac{1}{p^s} = [1 - \underset{i}{C}(s'\ln p)] + i\,\underset{i}{S}(s'\ln p) \tag{7}$$

Inverting this we arrive at,

$$\frac{1}{1-\frac{1}{p^s}} = \frac{1}{2}[1 - i\frac{\underset{i}{S}(s'\ln p)}{1-\underset{i}{C}(s'\ln p)}] = \frac{1}{2}[1 - i\frac{\underset{i}{S}(\frac{s'\ln p}{2}+\frac{s'\ln p}{2})}{1-\underset{i}{C}(\frac{s'\ln p}{2}+\frac{s'\ln p}{2})}]$$

$$= \frac{1}{2}[1 - i\frac{\underset{i}{C}(\frac{s'\ln p}{2})}{\underset{i}{S}(\frac{s'\ln p}{2})}] = \frac{1}{2}[1 - i\frac{1}{\underset{i}{T}(\frac{s'\ln p}{2})}]$$

We can then write;

$$F(p\ ) \equiv \frac{1}{2}[1 - i\frac{1}{\tan(\frac{s'\ln p}{2})}] \tag{8}$$

Equation (8) will be seen to reveal much of the hidden workings of the Zeta function. Before we can proceed to explain this, we must detour into the interesting world of hyperbolic geometry.

Brief introduction to a very useful Non-Euclidean geometry

There exists a vast literature on the non-Euclidean brand of geometry known as hyperbolic geometry [3]. The non-Euclidean geometry of Bolyai, Gauss, and Lobachevsky, or axiomatic hyperbolic geometry, is the system of geometry which rests logically on the basis of Euclidean geometry and an assumption contradicting Euclid's parallel postulate. This new postulate can be stated [3] as:

Straight lines *g, h* (Fig. I) exist which have a transversal such that *a* + *b* is less than two right angles and which do not meet.

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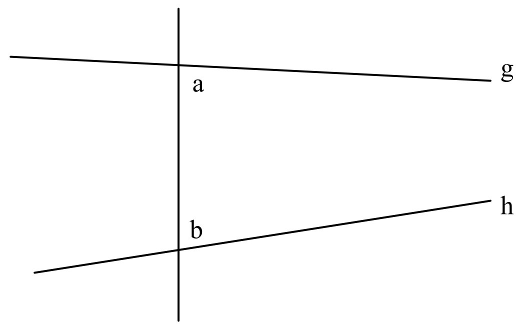

Fig. I In hyperbolic geometry, there exist angles *a* and *b*, both less than 90^o and lines *g* and *h* which do not meet.

The axiomatic approach to Hyperbolic geometry has revealed much about the nature of this space. We know in hyperbolic geometry there are two kinds of parallel lines, those with a common perpendicular and those without a common perpendicular.

It is the second kind of parallels we will be most interested in. In fact, equation (8) will be seen as equivalent to the defining formula for the parallel lines known as boundary parallels. This in turn will place constraints on allowable values of s.

To get a feel for this hyperbolic world we start by quoting without proof the following theorem of hyperbolic geometry referring to boundary parallels. (Refer to [3] for details):

Theorem 1: To every given segment PQ there corresponds two equal angles of parallelism and they are acute. Every given acute angle is an angle of parallelism corresponding to some segment. (Fig. 2)

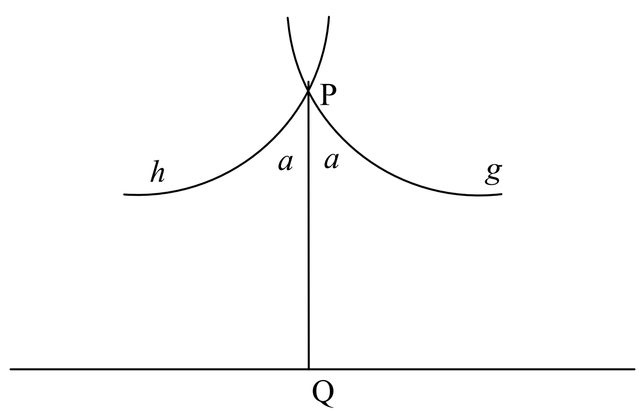

Fig.2 Line segment PQ and its two boundary parallels, g and h, both making acute angle *a* with the segment PQ.

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In hyperbolic space, the relation between the size of a segment and the magnitude of a corresponding angle of parallelism can be described numerically [3]. To do this we must choose a unit segment in terms of which to measure all segments and angles. (Length and angle measures only require a single chosen unit measure. Angle measures depend on the method of angular measurement devised, and are proportional to some chosen unit length in the space.)

The formula can be shown [3] to be:

$$\alpha = 2\tan^{-1}(e^{-\left[\frac{x}{k}\right]}) \qquad (9)$$

where k is an specified parameter.

When a particular value is assigned to k, the equation to which (9) reduces for that k value, gives the relation between x and α for some special unit segment. Thus, for k = 1, we obtain $\alpha = 2\tan^{-1}(e^{-x})$.
Since $\alpha = 2\tan^{-1}(e^{-1})$, the unit segment when x = 1, (and when k is specified as 1), is a segment whose corresponding angle of parallelism α is about 40^{o}.

We shall see that for any given k, there is always "a unit segment" which corresponds to a segment whose associated angle of parallelism is $\frac{\pi}{4}$. It is related to the 'natural measure' of the axis for any given k and its length is given as $(k\sinh^{-1}1)$. Its use simplifies work immensely. We can think of k as a scaling in hyperbolic space.

2. *Musès' r-ε model of hyperbolic space*

Various models have been proposed as devices to help conceptualize the axiomatic approach to hyperbolic geometry. Each model has its own has advantages and disadvantages.
In this author's opinion, Charles Musès' *r-ε model of hyperbolic space* far exceeds any other model proposed in versatility and elegance. Analogous to ordinary complex numbers, Muses' constructs a 2-Dimensional plane with a real axis and a second, new hypernumber axis, labeled ε, where epsilon is defined as $\varepsilon^2 = 1$ and $\varepsilon \neq 1$. ε is a proper square root of one, [4,]

According to Musès', 'Every geometry has an associated arithmetic and every arithmetic has an associated geometry'. [4]

Hyperbolic geometry is the associated geometry of the arithmetic of numbers of the form $w = a + \varepsilon b$, $a, b \in R$.

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Kevin Carmody's series of three Sedenions papers [5,6,7] provide a detailed introduction into the dynamics of the arithmetics and geometries composed of various specific combinations of these hypernumbers. Sobczyk [10] and Hucks [11] have also begun to explore the *r-ε hyperbolic* representation.
(Appendix B also provides a brief introduction to this space of numbers from a number theoretic point of view).

Analogous to ordinary *i-complex* numbers, *ε-complex* numbers can be polarized through the hyperbolic identity $e^{\varepsilon\theta} = \cosh\theta + \varepsilon\sinh\theta$.

$$\begin{aligned} w &= a + \varepsilon b \\ &= r(\cosh\theta + \sinh\theta) \text{ where } \tanh\theta = \frac{b}{a},\ r^2 = a^2 - b^2 = \|w\| \\ &= re^{\varepsilon\theta} \end{aligned} \tag{10}$$

In Figure 3 it can be seen that hyperbolic complex numbers with the same norm, form hyperbolas in r-ε space. The hyperbolas are asymptotic to two lines of zero divisors.

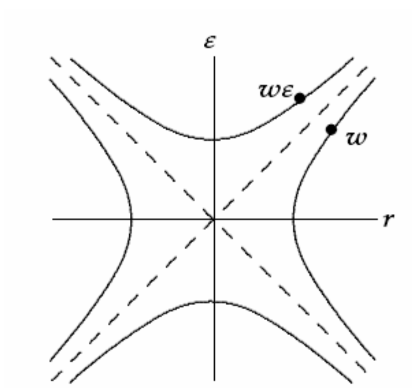

Fig. 3 Hyperbolic Complex numbers with the same Norm form Hyperbolas in r-ε space.

In *r-ε space* the two lines of zero divisors are given by the equation y = $k(1 \pm \varepsilon)$.
Multiplied together the product of conjugate zero divisors is zero. This particular property of hyperbolic numbers is generally not encountered in real or complex arithmetics, (the property is only associated with zero and *0+0i* in those arithmetics respecitively). We expect interesting behavior in hyperbolic space anytime curves cross these zero-divisors.

(Appendix B also introduces a generalized Euclidean algorithm with due consideration for the zero-divisors).

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3. Boundary Parallels in r- ε space.

So as to expedite our goal of drawing conclusions about the Zeta Function with complex argument in Hyperbolic Quaternion space, we now investigate further the formula given in equation (9) relating angle and distance in hyperbolic geometry. This will enable us to construct the defining functions in *r- ε space* relating to angles of parallelism. We will find a very simple and elegant representation in *r- ε space*,

Equation (9) is a relationship between angle and distance. It may be written in the following equivalent form [9]:

$$\tan\alpha = \frac{1}{\sinh\frac{x}{k}}, \tag{11}$$

where α is the angle of parallelism associated with the line segment of length x.

In normal Euclidean 2-D space, $\tan\alpha$ is often associated with the gradient of a line tangential to some point on a given curve. Applying this knowledge we may work backwards to derive the equation of a boundary parallel in *r- ε space.*

Labeling each point in *r- ε* space as *(x, y),* we define the Boundary Parallel function P(x) as:

$$\frac{dP(x)}{dx} \equiv \tan\alpha = \frac{1}{\sinh\frac{x}{k}} \tag{12}$$

By integrating equation (12) we arrive at:

$$P(x) = \int \frac{1}{\sinh\frac{x}{k}}\,dx = k\ln\left|\tanh\frac{x}{2k}\right| + C \tag{13}$$

The choice of the constant C enables us to fine tune our representation.
$P_0(x)$ will denote equation (13) with constant of integration C = 0.
Similarly $P_c(x)$ will refer to equation (13) with constant of integration C = c.
Figure 4 and 5 show the cases when C =0 and 2 respectively. We can see in Figure 4 that $P_0(x)$ is asymptotic to *y = 0* as x approaches infinity. It never crosses the x-axis in the finite realm.
In Figure 5, $P_2(x)$ we have translated $P_0(x)$ a distance 2 units in the positive y-axis direction.

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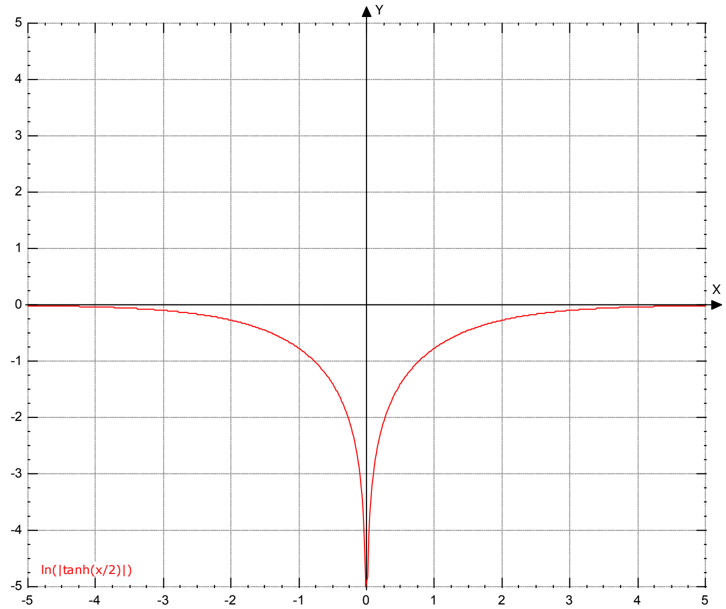

Fig. 4 Shows the plots of $P_0(x) = k\ln\left|\tanh\dfrac{x}{2k}\right|$, *k=1.*

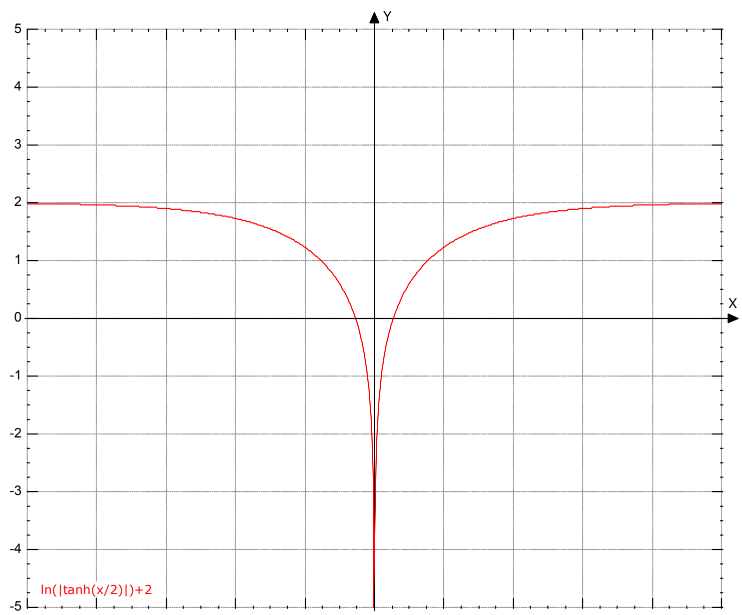

Fig. 5 The plot of $P_2(x) = k\ln\left|\tanh\dfrac{x}{2k}\right| + 2$, $k = 1$

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Denoting the *x-axis* intercepts of $P_c(x)$ as $\pm x_c$, and by translating $P_0(x)$ an amount

$$C = \left| k \ln \left| \tanh \frac{x_c}{2k} \right| \right|$$

in the positive y direction, we can position our curve so that it makes precisely the angle α with the x-axis at x_c. We have in effect associated the correct angle of parallelism for the given line segment x_c.

Our Boundary Parallel function then becomes:

$$P_{k,x_c}(x) = k \ln \left| \tanh \frac{x}{2k} \right| + k \ln \left| \tanh \frac{x_c}{2k} \right| \qquad (14)$$

(Note, written in polar form this equation becomes a statement about the argument of a hyperbolic number.)

3a. Useful Measures in Hyperbolic Space

The construction of Boundary Parallels in this hyperbolic representation suggests a natural unit measure for our axes:

For a given *k*, the segment whose corresponding angle of parallelism α is $\frac{\pi}{4}$, can be calculated from equation (9) as

$$\begin{aligned} x &= -k \ln(\tan \frac{\pi}{8}) \\ &= k \sinh^{-1} 1. \end{aligned} \qquad (15)$$

By marking our axes in 'units' of $\sinh^{-1} 1$, we know then that the Boundary Parallel $P_{k,x_c}(x)$, for a given *k* value AND which makes angle $\frac{\pi}{4}$ radians with the *x-axis*, will cross precisely at the value $x_c = k \sinh^{-1} 1$. Amongst all the *y*-translated *versions of a* boundary parallel for a given *k*, only one will make an angle of exactly 45 degrees with the x-axis, and the length of this segment is also equal to *k*, in units of $\sinh^{-1} 1$.

Let us now define a new method to measure the angle of a boundary parallel. Our familiar degree and radian angular measures are related to the circumference of a circle.

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We note in r- ε *space* the boundary parallels are all asymptotic to $y = -\infty$, in the direction of parallelism, and that the angles open to face the *y-axis*. If we draw a tangent, as in Figure 7, and associate the y-intercept of this tangent as the measure of the angle, then each boundary parallel, for a given k will be assigned a unique number, of absolute magnitude between 0 and ∞. In section 5 we will put this measure to good use.

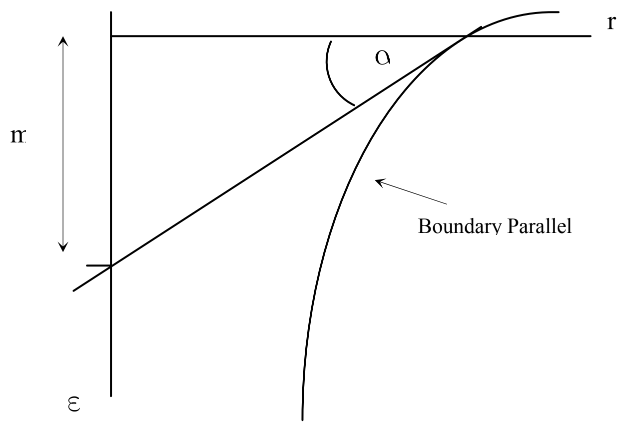

Figure 7 Graph showing two angle measures of the same boundary parallel, m and α .

We have one further introductory step to take before we can continue with our discussion of the Riemann Zeta function and confirm the conjecture. Importantly now, we must digress into the higher space of Hyperbolic Quaternions. Once introduced to Hyperbolic Quaternions, we will see that Equation (8) is really an invitation into a much larger world!

4. The Hyperbolic Quaternions

Charles Musès' [4] and Kevin Carmody [5,6,7] have developed detailed knowledge of the algebra and geometry of Hyperbolic Quaternions. (The reader is directed to [5,6,7] for a more complete introduction.)

Hyperbolic Quaternions are defined as the 4-Meta-dimensional hypernumbers based on 3 types of fundamental hypernumber units: 1, i and ε, (where ε is a proper square root of +1). Musès' term Meta-dimensional is used to emphasis the distinction between the different types of axis, as compared to the term 'multi-dimensional' which refers to multiple copies of the same type of axis (e.g. 3-D *real* space is composed of three copies of the r hypernumber unit: $R \times R \times R$)

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In Cartesian form, hyperbolic quaternions are written as

$$s = a + bi_1 + c\varepsilon_2 + d\varepsilon_3 \qquad (a,b,c,d \in \mathrm{R}) \tag{16}$$

The multiplication table for Hyperbolic Quaternions is defined as in Figure 8.

1	i_1	ε_2	ε_3
i_1	-1	ε_3	$-\varepsilon_2$
ε_2	$-\varepsilon_3$	1	$-i_1$
ε_3	ε_2	i_1	1

Figure 8. The Multiplication Table for Hyperbolic Quaternions

Interestingly, multiplication in this hypernumber space is associative but not commutative. It also contains nilpotents, numbers which when squared equal zero:
$(i_1 + \varepsilon_2)^2 = -1 + \varepsilon_3 - \varepsilon_3 + 1 = 0$.

Kevin Carmody has constructed an impressive looking polar form for hyperbolic quaternions [6]. His method will prove invaluable in the following work with the Zeta function.

Of particular interest is the following transformation used within his construction:

Consider the 2-Dimensional hyperbolic subspace ($\varepsilon_2, \varepsilon_3$). We can rewrite the numbers in this plane in the following polar form:

$$\begin{aligned} z &= c\varepsilon_2 + d\varepsilon_3 \\ &= r(\cos\phi + i_1 \sin\phi)\varepsilon_2 \\ &= re^{i\phi}\varepsilon_2 \end{aligned} \tag{17}$$

where $\tan\phi = \dfrac{d}{c}$ and $r^2 = c^2 + d^2$

We note the number $re^{i\phi}\varepsilon_2$ is a square root of +1 since $\left(re^{i\phi}\varepsilon_2\right)^2$

$$= \left(re^{i\phi}\varepsilon_2\right)\left(re^{i\phi}\varepsilon_2\right) = \left(\varepsilon_2 re^{-i\phi}\right)\left(re^{i\phi}\varepsilon_2\right) = \left(\varepsilon_2\right)^2 = +1$$

These roots lie on a circle, [5], as shown in Figure 9.

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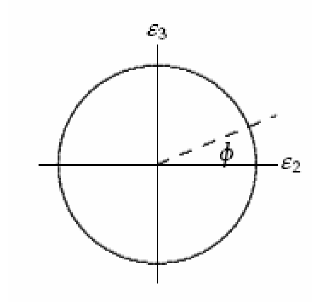

Figure 9 Square roots of +1 in the $\varepsilon_2 - \varepsilon_3$ plane.

Although the discussion has been brief, we have now introduced all the key elements necessary to discuss the Riemann Zeta conjecture and determine its validity. In a relatively few number of steps, we can derive a formula to calculate the zeros of the Zeta function.

5. *Confirming Riemann's Zeta Function Conjecture*

In Section One we wrote the function $F(p_j)$ in an odd looking trigonometric form and stated that this equation would reveal much about the inner workings of the Zeta function. We wrote Equation (8) as,

$$F(p\ ) \equiv \frac{1}{2}[1 - i\frac{1}{\tan(\frac{s'\ln p}{2})}] \qquad (8)$$

where p is prime and $s' = (si) = b - ai$

In Section One we also assumed that if a single value of the Infinite Product of Primes was equal to zero, then the whole Zeta function was also equal to zero.

Let us now set $F(p_j)$ equal to zero and investigate further.
With $F(p_j) = 0$, we can re-arrange terms to arrive at a very familiar looking hyperbolic function:

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Starting with;

$$0 = \frac{1}{2}[1 - i\frac{1}{\tan(\frac{s'\ln p}{2})}] \tag{18}$$

we get,

$$2\tan^{-1}(i) = s'\ln p \tag{19}$$

and utilizing the following observation;

$$i = e^{(-1)^n(2n+1)\frac{\pi}{2}i} \tag{20}$$

and writing $s'\ln p$ in polar form as;

$$s'\ln p = me^{i\phi} \tag{21}$$

where $m = \ln p\sqrt{a^2+b^2}$ and $\tan\phi = \frac{-a}{b}$,

Equation (19) then becomes;

$$2\tan^{-1}\left(e^{(-1)^n(2n+1)\frac{\pi}{2}i}\right) = me^{i\phi} \tag{22}$$

We observe that by setting our function $F(p_j)$ equal to zero, we have what looks similar to the defining formula between segment distance and associated boundary parallel angle in a hyperbolic space.

To make it such, one more observation will lead us directly to the formula we seek.

The numbers $e^{i_1\phi}\varepsilon_2$ were shown to be proper roots of 1. In reference to Figure 10, we can say the plane $\left(i_1, e^{i_1\phi}\varepsilon_2\right)$ is actually a hyperbolic plane.

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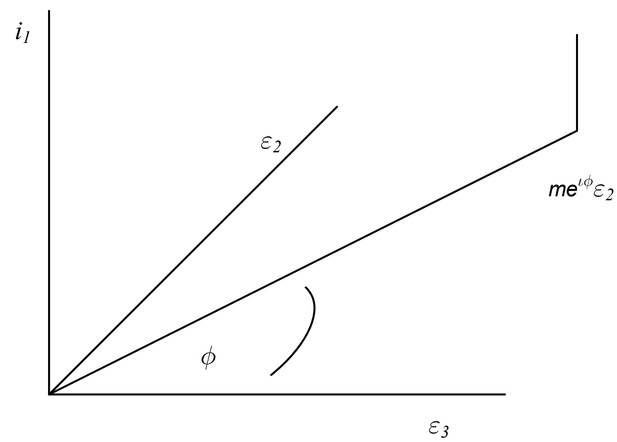

Figure 10 The hyperbolic plane $\left(i_1, e^{i_1\phi}\varepsilon_2\right)$.

To make measurements in the $\left(i_1, e^{i_1\phi}\varepsilon_2\right)$ hyperbolic plane, let us agree to use the unit measure $\sinh^{-1} 1$, as discussed in Section 3a. To measure angles in the $\left(i_1, e^{i_1\phi}\varepsilon_2\right)$ hyperbolic plane, let us use the tangent method defined in Section 3a, to assign a unique real number, m say, to our angles of parallelism, (for a given *k)*.

With these conventions, the general Boundary Parallel Formulas in the $\left(i_1, e^{i_1\phi}\varepsilon_2\right)$ hyperbolic plane becomes:

$$2\tan^{-1}\left(e^{\frac{-x\sinh^{-1}1}{k}}\right) = m\sinh^{-1}1$$

where k is a parameter. (23)

We will now show that $F(p_j) = 0$ implies a special case of the distance-angle formula belonging to the $\left(i_1, e^{i_1\phi}\varepsilon_2\right)$ hyperbolic plane.

Equation (20) specifies an infinite set of numbers:

$$i = e^{(-1)^n(2n+1)\frac{\pi}{2}i} \in \left(\frac{\pi}{2}, \frac{-3\pi}{2}, \frac{5\pi}{2}, \ldots\right).$$

We may consider these numbers as points along the *i-axis* in the $\left(i_1, e^{i_1\phi}\varepsilon_2\right)$ plane.

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Let us define the actual segment sizes they correspond to as

$$(-1)^n (2n+1)\frac{\pi}{2} \times \sinh^{-1} 1 .$$

(We might just as easily say the segment length is just $(-1)^n (2n+1)\frac{\pi}{2}$, for a given n)

Similarly, $s' \ln p = me^{i\phi}$ may be considered a point situated a distance *m* along the $e^{i_1\phi}\varepsilon_2$ axis, and which we will say corresponds to a segment length of $m \times \sinh^{-1} 1$.

With these identifications we can now say that the condition $F(p_j) = 0$ implies the following special case of the distance-angle formula belonging to the $\left(i_1, e^{i_1\phi}\varepsilon_2\right)$ hyperbolic plane:

$$2\tan^{-1}\left(e^{-\left(\frac{(-1)^n (2n+1)\frac{\pi}{2}\sinh^{-1} 1}{k}\right)} \right) = m \sinh^{-1} 1$$

$$\text{where } m = \ln p\sqrt{a^2+b^2} \quad \text{and} \quad k \text{ is unspecified,} \tag{24}$$

5a Calculating the Zeros of the Zeta Function with complex arguments

For a given value of *s*, Equations (24), can be re-arranged into a formula for b:

$$b = \left(\left(\frac{2\tan^{-1}\left(e^{-\left(\frac{(-1)^n (2n+1)\frac{\pi}{2}\times\sinh^{-1} 1}{k}\right)} \right)}{\sinh^{-1} 1 \times \ln p} \right)^2 - a^2 \right)^{\frac{1}{2}} \tag{25}$$

p prime, *n* integer, *k* a real-valued parameter.

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Riemann's conjecture is that all 'interesting' zeros of the Zeta function with complex argument lay on a line in the complex *(r,i)* plane with real part a = ½.

Observing Equation (25) we can say these are actually just a few of the many interesting zeros. We can see the Norm of the complex number is more fundamental, as is the specification of the parameter *k.*

Values of particular interest for *b* appear to be the asymptotic values of *k* approaching zero and the infinities.

6 Summary

The present work rests heavily on the assumption that an infinite product, with all factors non-zero except one, will multiply to equal to zero. The trigonometric form we used for the Zeta factor was also derived on the condition of non-zero divisors. Assuming these assumptions hold, we have been able to show that the zero valued Zeta Function corresponds to a special case of the distance-angle formula belonging to the $\left(i_1, e^{i_1\phi}\varepsilon_2\right)$ hyperbolic plane of Hyperbolic Space.

More complete investigations of the infinite and infinitesimal and zero divisors are continuing. Appendix B discusses further the notion of zero divisors in *r-ε* space.

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Appendix A *Generalized Sine and Cosine Functions:*

In hypernumber spaces that are associative, we can generalize our familiar notions of trigonometric functions in a very straight forward manner:

Let A and B be hypernumbers of the form $m u_i$ where u_i is one of the hypernumber units $1, i, \varepsilon$ or the mixed hypernumber $i_o \equiv i_n \varepsilon_n$ and *m* is real.

Recalling our familiar trig definitions:

$$\cosh(A) \equiv \frac{e^A + e^{-A}}{2} \qquad sinh(A) \equiv \frac{e^A - e^{-A}}{2}$$

$$\cos(A) \equiv \frac{e^{iA} + e^{-iA}}{2} \qquad \sin(A) \equiv \frac{e^{iA} - e^{-iA}}{2i} \tag{A1}$$

we define more generally:

$$\underset{B}{C}(A) = \frac{e^{BA} + e^{-BA}}{2} \quad \text{and} \quad \underset{B}{S}(A) = \frac{e^{BA} - e^{-BA}}{2B} \tag{A2}$$

It is important to note that in commutative spaces we can arrange the summands in the exponential powers with impunity:

$$e^{a+bi} = e^a e^{bi} = e^{bi+a}$$

Equation (44) enables us to write the identities:

$$e^{BA} = \underset{B}{C}(A) + B\underset{B}{S}(A) \tag{A3}$$

Substituting *B*= *i* and the more familiar A=θ, equation (A3) can be seen as a generalization of the complex exponential function:

$$e^{i\theta} = \cos\theta + i\sin\theta$$

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In this generalized terminology the following familiar trig functions are written as:

$$\underset{1}{C}(\theta) \equiv cosh(\theta) \qquad \underset{1}{S}(\theta) \equiv sinh(\theta)$$

$$\underset{i}{C}(\theta) \equiv cos(\theta) \qquad \underset{i}{S}(\theta) \equiv sin(\theta) \qquad \text{(A4)}$$

Note also the following identities:

$$\underset{\varepsilon}{C}(\theta) = \frac{e^{\varepsilon\theta} + e^{-\varepsilon\theta}}{2} = \frac{1}{2}(\cosh\theta + \varepsilon\sinh\theta + \cosh\theta - \varepsilon\sinh\theta) = \cosh\theta$$

$$\underset{\varepsilon}{S}(\theta) = \frac{e^{\varepsilon\theta} - e^{-\varepsilon\theta}}{2\varepsilon} = \frac{1}{2\varepsilon}(\cosh\theta + \varepsilon\sinh\theta - \cosh\theta + \varepsilon\sinh\theta) = \sinh\theta$$

$$\underset{\varepsilon}{C}(\theta i) = \cos\theta \qquad \underset{\varepsilon}{S}(\theta i) = i\sin\theta$$

$$\underset{\varepsilon}{C}(\theta\varepsilon) = \cosh\theta \qquad \underset{\varepsilon}{S}(\theta\varepsilon) = \varepsilon\sin\theta$$

$$\underset{\varepsilon}{C}(\theta i_0) = \cos\theta \qquad \underset{\varepsilon}{S}(\theta i_0) = i_0\sin\theta \qquad \text{(A5)}$$

We can construct definitions and identities similar to our familiar to trig functions:

$$\underset{\varepsilon}{T}(\theta) = \frac{\underset{\varepsilon}{S}(\theta)}{\underset{\varepsilon}{C}(\theta)} = \tanh(\theta)$$

$$\underset{\varepsilon}{C}(a+b) = \underset{\varepsilon}{C}(a)\underset{\varepsilon}{C}(b) + \underset{\varepsilon}{S}(a)\underset{\varepsilon}{S}(b)$$

$$\underset{\varepsilon}{S}(a+b) = \underset{\varepsilon}{S}(a)\underset{\varepsilon}{C}(b) + \underset{\varepsilon}{S}(a)\underset{\varepsilon}{C}(b) \qquad \text{(A6)}$$

Utilizing Kevin Carmody's polarizations [6], we note then we may write the exponential of a conic quaternion as either a product of four different types of generalized trig functions all with real arguments or as a product of a single type of trig ε functions but with differing types of hypernumber arguments:

$$e^{a+bi+c\varepsilon+di_o} = e^{(a\varepsilon+bi_o+c+di)\varepsilon}$$

$$= [\underset{1}{C}(a) + \underset{1}{S}(a)][\underset{i}{C}(b) + i\,\underset{i}{S}(b)][\underset{\varepsilon}{C}(c) + \varepsilon\,\underset{\varepsilon}{S}(c)][\underset{i_o}{C}(d) + i_0\,\underset{i_0}{S}(d)] \qquad \text{(A7)}$$

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$$=[\underset{\varepsilon}{C}(a\varepsilon_1)+\varepsilon_1\underset{\varepsilon}{S}(a\varepsilon_1)][\underset{\varepsilon}{C}(bi_0)+\varepsilon_1\underset{\varepsilon}{S}(bi_0)][\underset{\varepsilon}{C}(c)+\varepsilon_1\underset{\varepsilon}{S}(c)][\underset{\varepsilon}{C}(di_1)+\varepsilon_1\underset{\varepsilon}{S}(di_1)] \quad \text{(A8)}$$

and so we can construct general expansions of the form:

$$= Z_1 + Z_2 i_1 + Z_3 \varepsilon_1 + Z_4 i_0$$
$$= M_1 + \varepsilon_1 M_2 \quad \text{(A9)}$$

As noted previously, the generalized trigonometric functions introduced here are useful tools in the hypernumber spaces in which associativity and commutativity operate such as Conic Quaternions[5].

Hyperbolic Quaternions are associative but non-commutative. Further generalizations of trig-like functions will be necessary for further investigations in this space based on these types of functions.

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Appendix B *(r-ε) Space from a Number Theoretic point of view.*

In analogy to Gaussian Integers, we briefly investigate the Hyperbolic Integers.

1. Hyperbolic Integers

Define a general hyperbolic integer, Z [ε], as a number of the form

$$s = a + b\varepsilon \quad (a, b \in Z \text{ and } \varepsilon^2 = 1, \text{ where } \varepsilon \neq \pm 1). \tag{B1}$$

Analogous to ordinary Complex numbers, define the Conjugate of a hyperbolic integer

$s = a + \varepsilon b$

as

$$\bar{s} = a + \varepsilon b, \tag{B2}$$

and the *Norm* $N(s)$ of a hyperbolic integer as the product;

$$N(s) = (a + \varepsilon b)(a - \varepsilon b) = a^2 - b^2 \tag{B3}$$

Geometrically, the Norm of general hyperbolic numbers with $(a, b \in R)$, form hyperbolas in the r-ε plane, and are asymptotic to the line of zero divisors $\mathrm{y} = \mathrm{k}(1 \pm \varepsilon)$, (recall Fig 4). Hyperbolic integers form lattice points associated with these hyperbolas.

In ordinary *i-complex* numbers, the Modulus function is defined as the square root of the Norm function, $(N(s))^{\frac{1}{2}} = \sqrt{a^2 + b^2}$, and the Length function, which is a measure of the distance a point is from the origin, is defined as equal to $\sqrt{a^2 + b^2}$. Both function in *i-complex* numbers. have the same form and value.

In hyperbolic space however, the Modulus function and the Length functions are different and have distinct values, (except of course when the ε part is zero valued):

$$M(s) = \sqrt{a^2 - b^2}$$
$$L(s) = \sqrt{a^2 + b^2} \tag{B4}$$

We shall use the Norm of Hyperbolic Integers in the following discussion. Although the Norm of every Hyperbolic integer is an integer, it is not true that every integer is a norm. The integer 2 is a case in point, since this integer cannot be written in the form $a^2 - b^2$.

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We state the following theorems about hyperbolic integers and leave the proofs to the reader.

It can easily be shown the Norm is multiplicative: for α and $\beta \in Z[\varepsilon]$

$$N(\alpha\beta) = N(\alpha)N(\beta) \tag{B5}$$

Furthermore the only Hyperbolic integers which are invertible in $Z[\varepsilon]$ are ± 1 and $\pm\varepsilon$. Knowing a Hyperbolic Integer up to multiplication by a unit, is analogous to knowing an integer up to its sign.

2. Divisibility of the Hyperbolic Integers

Ordinarily, with our familiar complex numbers, we say β divides α (and write $\beta \mid \alpha$) if we can write $\alpha = \beta\gamma + 0$ for some $\gamma \in Z[i]$. In this case we say β is a divisor or a factor of α.

Divisibility in $Z[\varepsilon]$ can be defined in a similar way, but we must extend this definition in the following manner which takes into consideration that zero divisors in r-ε space also have Norm equal to zero.(Section 5 discusses Zero divisors further)

When we write $\beta \mid \alpha$ we will mean;

$\alpha = \beta\gamma +$ N(remainder)

where N(remainder) $\in n(1 \pm \varepsilon)$ AND N(remainder) = 0. (B6)

Theorem 2.1 A hyperbolic integer $\alpha = a + b\varepsilon$ is divisible by an ordinary 'real' integer c if $c \mid a$ and $c \mid b$ in Z.

To verify divisibility in $Z[\varepsilon]$ we use the familiar procedure of rationalizing the denominator and then testing the ratio.

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3. The Division Theorem

Hyperbolic numbers have an analogous Division Theorem to the real integer division theorem.

Theorem 3.1 (Division Theorem) For $\alpha, \beta \in Z[\varepsilon]$ with $N(\beta) \neq 0$, there are γ, r $\in Z[\varepsilon]$ such that

$\alpha = \beta\gamma + r$

$$\text{and } \frac{-N(\beta)}{2} \leq N(r) \leq \frac{N(\beta)}{2} .$$

Proof:

$$\text{Write } \frac{\alpha}{\beta} = \frac{\alpha}{\beta}\frac{\overline{\beta}}{\overline{\beta}} = \frac{\alpha\overline{\beta}}{N(\beta)} = \frac{m+n\varepsilon}{N(\beta)} \qquad (N(\beta) \neq 0)$$

and where we have set $\alpha\overline{\beta} = m + n\varepsilon$.

Choose $q_1, q_2 \in Z$ such that

$$0 \leq \left|\frac{m}{N(\beta)} - q_1\right| \leq \frac{1}{2} \quad \text{and} \quad 0 \leq \left|\frac{n}{N(\beta)} - q_2\right| \leq \frac{1}{2} .$$

Then

$$\left|\frac{\alpha}{\beta} - (q_1 + q_2\varepsilon)\right|^2 = \left|\frac{m+n\varepsilon}{N(\beta)} - (q_1 + q_2\varepsilon)\right|^2 = \left(\frac{m}{N(\beta)} - q_1\right)^2 - \left(\frac{n}{N(\beta)} - q_2\right)^2 \to A^2 - B^2$$

Now if $A^2 = 0$ and $B^2 = \frac{1}{2}$ or $A^2 = \frac{1}{2}$ and $B^2 = 0$, then

$$-\frac{1}{2} \leq \left|\frac{\alpha}{\beta} - (q_1 + q_2\varepsilon)\right|^2 \leq \frac{1}{2} .$$

So we can write:

$-\frac{1}{2}|\beta|^2 \leq |\alpha - \beta(q_1 + q_2\varepsilon)|^2 \leq \frac{1}{2}|\beta|^2$ since squared absolute values equal norms on $Z[\varepsilon]$,

where $\gamma = q_1 + q_2$ and we set $r = \alpha - \beta\gamma$.

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Given α and β we can use this algorithm to find γ as follows: First we rationalize the denominator and then select integer values for q_1 and q_2 by selecting the closest integers. As an example let

$\alpha = 11 + 3\varepsilon$ and $\beta = 1 + 8\varepsilon$

Rationalizing the denominator using conjugation;

$$\frac{\alpha}{\beta} = \frac{\alpha}{\beta}\frac{\overline{\beta}}{\overline{\beta}} = \frac{(11+3\varepsilon)(1-8\varepsilon)}{-63} = \frac{13}{63} + 1\frac{23}{63}\,\varepsilon$$

Now by selecting the nearest integers we arrive at

$0+1\varepsilon = q_1 + q_2\varepsilon$

$$\Rightarrow \text{r} = \alpha - \beta\gamma = 3 + 2\varepsilon.$$

And so we may write

$11 + 3\varepsilon = (1 + 8\varepsilon)(0 + \varepsilon) + (3 + 2\varepsilon).$

Checking, we see that $N(3 + 2\varepsilon) = 9 - 4 = 5$

and $-\frac{1}{2}N(\beta) \leq N(3 + 2\varepsilon) \leq \frac{1}{2}N(\beta)$

$\frac{-63}{2} \leq 5 \leq \frac{63}{2}.$

4. The Euclidean Algorithm

Let us take a closer look at divisors in $Z[\varepsilon]$.

As we noted in Section 2, we can more suggestively write $\beta \mid \alpha$ as

$$\alpha = \beta\,\gamma + \text{N(remainder)}$$

where N(remainder) is the Norm of the remainder and equaled zero there.

For Z and Z[i] the only numbers that have a norm of zero is 0 and 0 + 0i respectively. As a consequence of this, division in Z and Z[i] can always be simplified and written in the form

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$$\alpha = \beta\ \gamma + 0 = \ \beta\ \gamma$$
$$\alpha = \beta\ \gamma + (0 + 0i) = \ \beta\ \gamma \qquad \text{(B7)}$$

without any confusion as to what the zero remainder is.

In hyperbolic space however, there are an infinite number of Norm zero numbers, all of the form $n(1+\varepsilon)$, each located on one of the lines y = $\pm\ n(1+\varepsilon)$

Accordingly, we generalize our notion of divisors in hyperbolic space as follows:

Definition 4.1 For α , $\beta \in$ Z[ε], if $\beta \,|\, \alpha$ then we can write

$$\alpha = \beta\ \gamma + \text{N(Remainder)}$$
$$\text{s.t. N(Remainder) = 0.} \qquad \text{(B8)}$$

For example,

Let α = 11 + 3 ε and β = 3 + 2 ε

then $\alpha = \beta\ \gamma + \text{N(Rem)} \Rightarrow$
$$11 + 3\varepsilon = (3 + 2\varepsilon)(5 - 3\varepsilon) + (2 + 2\ \varepsilon)$$

and N(rem) = N(2 + 2 ε) = 0

and we say β divides α .

Definition 4.2 α and β have a common divisor γ if for some r, m $\in$ Z[ε], we can write

α = m γ + r β = m γ + r where N(r) = 0

(Equivalently we may write

α = r mod m and β = r mod m and N(r) = 0)

Definition 4.3 For non-zero α , $\beta \in$ Z[ε], a greatest common divisor γ_{GCD} of α and β is a common divisor with maximal Norm.
i.e $N(\gamma) \leq N(\gamma_{GCD})$ for all common divisors, γ , of α and β .

If γ_{GCD} is a greatest common divisor of α and β , so are its unit multiples

$-\gamma_{GCD}$ $\qquad$ $\varepsilon\ \gamma_{GCD}$ $\qquad$ $-\varepsilon\ \gamma_{GCD}$

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Definition 4.4 When α and β only have unit factors in common, we shall say α and β are relatively prime.

Theorem 4.5 (Euclid's Algorithm) Let α , β $\in$ Z [ε] with N(α), N(β) $\neq$ 0. Recursively apply the division theorem, starting with the pair α , β and make the divisor and remainder in one equation the new dividend and divisor in the next, provided the remainder does not have Norm equal to zero:

$$\alpha = \beta\gamma_1 + r_1 \qquad \frac{-N(\beta)}{2} \le N(r_1) \le \frac{N(\beta)}{2}$$

$$\beta = r_1\gamma_2 + r_2 \qquad \frac{-N(r_1)}{2} \le N(r_2) \le \frac{N(r_1)}{2}$$

$$r_1 = r_2\gamma_2 + r_3 \qquad \frac{-N(r_2)}{2} \le N(r_3) \le \frac{N(r_2)}{2}$$

•

•

•

The last non-zero Norm remainder is divisible by all common divisors of α and β , and is itself a common divisor, so it is a greatest common divisor of α and β .
If the last non-zero Norm remainder is a unit factor, then α and β are relatively prime.
The algorithm stops once a Norm zero remainder is encountered.

As an example, let

α = 32 + 10 ε
β = 4 + 11 ε .

Applying the division algorithm, we first use the conjugate:

$$\frac{\alpha}{\beta} = \frac{\alpha}{\beta}\frac{\overline{\beta}}{\overline{\beta}}$$

Proceeding from here:

$\alpha = \beta\gamma_1 + r_1$ is

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$32 + 10\varepsilon = (4 + 11\varepsilon)(0 + 3\varepsilon) + (-1 - 2\varepsilon)$ and $\frac{-N(4 + 11\varepsilon)}{2} \le N(-1-2\varepsilon) \le \frac{N(4 + 11\varepsilon)}{2}$

and continuing,

$4 + 11\varepsilon = (-1-2\varepsilon)(-6 + \varepsilon) + (0 + 0\varepsilon)$ and $N(0 + 0\varepsilon) = 0$

In this case GCD is $(-1-2\varepsilon)$.

Checking that $(-1-2\varepsilon)$ is indeed a divisor of both α and β:

$\alpha = (-1-2\varepsilon)(4-18\varepsilon) = 32 + 10\varepsilon$
$\beta = (-1-2\varepsilon)(-6 + \varepsilon) = 4 + 11\varepsilon$

as can be easily verified.

5 Zero Divisors and Infinities

According to the definition given in equation (B3), the Norm of the hyperbolic numbers of the form $k(1+\varepsilon)$ will be;

$$N\left(k(1+\varepsilon)\right) = k^2(1+\varepsilon)(1-\varepsilon) = 0. \qquad \text{(B9)}$$

Because their Norm equals zero, hyperbolic conjugate pairs of the form $k(1+\varepsilon)$ might be thought of as factors of zero, or zero divisors.

Just as with the Real numbers where we associate an infinite value to the reciprocal of zero, that is $\frac{1}{0} \equiv \infty$, we can similarly conceive of $\frac{1}{(1 \pm \varepsilon)}$, with its 'norm zero denominator', as a type of non-real infinite value in the ε space. Each number of the form $\frac{1}{k(1+\varepsilon)}$, can be considered as a different number in ε infinity. Geometrically their values are all infinitely far in the $(1 \pm \varepsilon)$ direction of the hyperbolic numbers. [12]

The appearance of any of these 'norm zero denominator' numbers in finite hyperbolic arithmetic, implies we are working with mixed arithmetics: finite and ε infinity arithmetic. Infinite arithmetic is different from finite arithmetic. To show this very simply, we provide two examples. Furthermore, we leave as open questions just what the rules of infinite arithimetics are.

Without proper handling of these 'norm zero denominator' infinities, contradictions will appear in finite r-ε arithmetic. The term 'proper handling', implies in part, that we use

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conjugation, just as we did with the finite hyperbolic numbers, to apply a kind of 'rationalization' of our infinities, before we do any other arithmetic operation. However, this is not everything as we shall see.

Let us define:

$$\frac{1}{(1+\varepsilon)} \equiv \frac{1}{(1+\varepsilon)} \times \frac{(1-\varepsilon)}{(1-\varepsilon)} = \frac{(1-\varepsilon)}{0} = \frac{1}{0} + \frac{-1}{0}\varepsilon \qquad \text{(B10)}$$

We provide two examples of apparent contradictions in finite r-ε arithmetic that reveal the unrecognized influence of infinite arithmetic:

Example 1

If we assume $\frac{1}{(1+\varepsilon)}$ can be written in the general form $a + b\varepsilon$, where $a, b \in$ finite R, then an apparent contradiction arises by assuming we can easily simplify each side of the following equation by multiplying by $(1+\varepsilon)$:

$$\frac{1}{(1+\varepsilon)} = \mathrm{a} + \mathrm{b}\varepsilon$$

$$1 = (1+\varepsilon)(\mathrm{a} + \mathrm{b}\varepsilon)$$

$$1 + 0\varepsilon = (\mathrm{a} + \mathrm{b}) + (\mathrm{a} + \mathrm{b})\,\varepsilon\,.$$

Equating real and hyperbolic parts implies the contradiction a + b = 1 AND a + b = 0.

If however we first conjugate the 'zero norm denominator' type infinity, we write the LHS as in Equation (B10) as:

$$\frac{1}{(1+\varepsilon)} \times \frac{(1-\varepsilon)}{(1-\varepsilon)} = \frac{(1-\varepsilon)}{0} = \frac{1}{0} + \frac{-1}{0}\varepsilon = a + b\varepsilon$$

Now when we equate real and hyperbolic parts we see that *a* and *b* are infinite, and the contradiction has disappeared from the finite arithmetic: there are no finite solutions!

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Example 2

In this example, finite $r-\varepsilon$ arithmetic appears non-associative when zero divisors are involved and we apply finite arithmetic rules to simplify:

$$\left[\varepsilon\times(1+\varepsilon)\right]\times\frac{1}{(1+\varepsilon)}\neq\varepsilon\times\left[(1+\varepsilon)\times\frac{1}{(1+\varepsilon)}\right]$$

$$1\neq\varepsilon$$

However, if we conjugate first our "Zero Norm Denominator" type infinities, we arrive at:

LHS

$$\begin{aligned}
&=\left[\varepsilon\times(1+\varepsilon)\right]\times\frac{1}{(1+\varepsilon)}\\
&=\left[1+\varepsilon\right]\times\left(\frac{1}{0}+\frac{-1}{0}\varepsilon\right)\\
&=\frac{1}{0}+\frac{-1}{0}\varepsilon+\frac{1}{0}\varepsilon+\frac{-1}{0}\varepsilon^2\\
&=0
\end{aligned}$$

RHS

$$\begin{aligned}
&=\varepsilon\left[(1+\varepsilon)\times\frac{1}{(1+\varepsilon)}\right]\\
&=\varepsilon\times\left[(1+\varepsilon)\times\left(\frac{1}{0}+\frac{-1}{0}\varepsilon\right)\right]\\
&=\varepsilon\left[\frac{1}{0}+\frac{-1}{0}\varepsilon+\frac{1}{0}\varepsilon+\frac{-1}{0}\varepsilon^2\right]\\
&=0
\end{aligned}$$

Implying that $\frac{(1+\varepsilon)}{(1+\varepsilon)}=0\neq1$.

It would seem then to maintain certain rules of finite arithmetic we are either to assume infinite arithmetic here is non-associative or give up the commutative notion of real and infinite numbers together. Research into infinite sums such as the alternating harmonic series suggests these are not unwarranted assumptions. [13].

We have been making distinctions between different types of infinities, based on whether they are real numbers, *i-complex* numbers or ε*-complex* numbers, and we have further made the distinction in ε-hyperbolic numbers of 'Zero Norm Denominator' type infinities.

Musès' [8] informs us that zero and infinity are merely labels for whole sets of numbers that are really only zero (or infinite) with respect to the finite numbers.

In this sense, 0 and ∞ are only symbols used to approximate infinitesimally small or infinitely large numbers. These numbers are generally beyond the manageability of the

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chosen representation schema for the numbers of the algebraic system. Generally, every number smaller than a certain limit is referred to as zero and every number larger than a certain limit are associated with the infinity symbol. In this sense zero and infinity can be considered multi-valued.

These sets of infinitesimals and infinite numbers however, do have unique values which are different from just zero '0' or infinity '∞', and they can be manipulated to influence the real arithmetic. A notable example is $\frac{\sinh x}{x}$ which appears as the so-called non-determinant form $\frac{0}{0}$ when numerator and denominator are each worked out separately, but which approaches the value 1 as $x \to 0$ in its infinite series representation. It's true nature being revealed once the appropriate representation is used. A proper arithmetic of these numbers is still forthcoming.

Since Zero '0' and Infinity '∞'are both really 'multi-valued' and multi-typed (recall our use of the different hypernumber types r, i, ε), we must be careful when working with the infinite and infinitesimals. We could define a symbolic system that is more sensitive than just applying a single label for zero or infinity. This may however add heavily to required time and space resources in performing calculations in the arithmetic. Sometimes a change of representation itself can influence our ability to interact with 'zero' and 'infinity'. (See Appendix C)

We should note in hyperbolic geometry, infinity and zero comes very much to the fore. Figure 4 shows that the Boundary Parellel curve $P_0(x) = k \ln \left| \tanh \frac{x}{2k} \right|$ is asymptotic to $x = \pm\infty$ and $y = -\infty$. Also, the line of zero divisors $y = k(1 \pm \varepsilon)$ cut a path across boundary parallels. There unique influence is surely to be felt.

Boundary Parallels can be considered as a kind of Cissoid of Diocles, each with their associated circle having radius located somewhere at infinity. The boundary parallels may also be conceived of as the evolutes of a parabolas whose vertices are situated somewhere else in infinity.

As Musès' discusses in [8], the defining property of infinity is not so much the size of its constituent members, but rather our not being able to order the individual members and reference them in the same way as we do for the finite numbers.

In future work, we aim to answer such questions as "At which point at infinity is our parabola vertex positioned? and "Which point at infinity should we locate the radii for our Cissoid formation circle?"

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Appendix C *A New Conjecture*

It seems appropriate to finish this discussion with a conjecture.

We have shown that by discussing the Zeta Function with complex argument in the larger representation of Hyperbolic Geometry, (which itself is a sub-space of Musès' 16-D Sedenions [5,6,7] we see that new insights immediately present themselves.

It has been a constant theme throughout this work that representation is everything.

Musès, in [4], states that 'Every geometry has an associated arithmetic and every arithmetic has an associated geometry'.

It can be added further that, how we choose to represent the numbers of those arithmetics determines to a large degree the properties of those geometries which we become aware of.

Katài & Szabò [13], Gubareni & Sinkov [14], Gilbert [15] and Akiyama, Brunotte, Pethő &Thuswaldner [16,17] have done pioneering work in this field with *real, i-complex and* ε*-complex numbers*. However, application of Muses' full stable of hypernumbers is yet to be fully utilized [4].

Representations, in a real sense, determine our interactions with not only the finite, but the zero and infinite as well.

We present the following Conjecture:

The theory of general relativity in tensor form, being a geometrical theory, is in fact a non-standard representation of the numbers of an unspecified arithmetic. Gravitational calculations correspond to manipulating the numbers in the arithmetic of that yet to be specified number system.

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